AUTOMORPHISMS OF THE TORELLI COMPLEX AND THE COMPLEX OF SEPARATING CURVES

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ABSTRACT. We compute the automorphism groups of the Torelli complex and the complex of separating curves for most of compact orientable surfaces. As an application, we show that the commensurators of the Torelli group and the Johnson kernel for such surfaces are naturally isomorphic to the extended mapping class group.

1. INTRODUCTION

Let S be a connected, compact and orientable surface with the Euler characteristic $\chi(S)$ negative. It is widely known that the complex of curves, $\mathcal{C}(S)$, for S plays an important role in the study of the mapping class group $\mathrm{Mod}(S)$ of S. In fact, understanding of the automorphism group of $\mathcal{C}(S)$ leads to the computation of the commensurator of $\mathrm{Mod}(S)$ as discussed in [16] and [22] (see also [24] for the automorphism group of $\mathcal{C}(S)$). The aim of this paper is to compute the automorphism group of the Torelli complex $\mathcal{T}(S)$ and the complex of separating curves, $\mathcal{C}_s(S)$, for most of surfaces S. These simplicial complexes are variants of the complex of curves and were introduced in [9] and [8] for closed surfaces to compute the commensurators of the Torelli group $\mathcal{I}(S)$ and the Johnson kernel $\mathcal{K}(S)$ for S (see also [25] for a related work). We refer to Section 2 for a precise definition of these complexes and groups. The extended mapping class group $\mathrm{Mod}^*(S)$ of S naturally acts on $\mathcal{T}(S)$ and $\mathcal{C}_s(S)$ by simplicial automorphisms. These actions induces the natural homomorphisms from $\mathrm{Mod}^*(S)$ into the automorphism groups $\mathrm{Aut}(\mathcal{T}(S))$ and $\mathrm{Aut}(\mathcal{C}_s(S))$, which are shown to be isomorphisms in the following theorems. As an application, we prove that the natural homomorphisms from $\mathrm{Mod}^*(S)$ into the commensurators of $\mathcal{I}(S)$ and $\mathcal{K}(S)$ are isomorphisms.

We shall recall the definition of the commensurator of a group Γ. Let $F(\Gamma)$ be the set of all isomorphisms between finite index subgroups of Γ. We say that two elements f, g of $F(\Gamma)$ are equivalent if there exists a finite index subgroup of Γ on which f and g are equal. The composition of two elements $f\colon \Gamma_1 \to \Gamma_2$, $g\colon \Lambda_1 \to \Lambda_2$ of $F(\Gamma)$ given by $f \circ g\colon g^{-1}(\Gamma_1 \cap \Lambda_2) \to f(\Lambda_2 \cap \Gamma_1)$ induces the product operation on the quotient set of $F(\Gamma)$ by this equivalence relation. This makes it into the group called the *(abstract) commensurator* of Γ and denoted by $\mathrm{Comm}(\Gamma)$. One has the natural homomorphism $\mathbf{i}\colon \Gamma \to \mathrm{Comm}(\Gamma)$ defined by inner conjugation. It is easily verified that $\mathbf{i}$ is injective if and only if the center of any finite index subgroup of Γ is trivial.

Let $S = S_{g,p}$ denote a connected, compact and orientable surface of genus g with p boundary components. Unless otherwise stated, we always assume a surface to satisfy these conditions.

Theorem 1.1. *Let $S = S_{g,p}$ be a surface and assume one of the following three conditions: $g = 1$ and $p \geq 3$; $g = 2$ and $p \geq 2$; or $g \geq 3$ and $p \geq 0$. Then*

(i) *the homomorphism from $\mathrm{Mod}^*(S)$ into $\mathrm{Aut}(\mathcal{T}(S))$ is an isomorphism.*
(ii) *the homomorphism $\mathbf{i}\colon \mathrm{Mod}^*(S) \to \mathrm{Comm}(\mathcal{I}(S))$ defined by conjugation is an isomorphism.*

When the genus of S is equal to one, the Torelli group for S is closely related to the braid groups on the torus. Following the argument of [23], we describe the commensurators of them in Section 7. An analogous result on the complex of separating curves and the Johnson kernel for S is stated as follows.

Theorem 1.2. *Let S be a surface satisfying the assumption in Theorem 1.1. Then*

(i) *the homomorphism from $\mathrm{Mod}^*(S)$ into $\mathrm{Aut}(\mathcal{C}_s(S))$ is an isomorphism.*
(ii) *the homomorphism $\mathbf{i}\colon \mathrm{Mod}^*(S) \to \mathrm{Comm}(\mathcal{K}(S))$ defined by conjugation is an isomorphism.*

The conclusion of these two theorems for closed surfaces is due to Brendle-Margalit [8]. Our proof of these theorems for surfaces of genus at least two partly follows their argument using sharing pairs and spines. When the genus of S is equal to one, there exists no sharing pair in S. In the case of $S_{1,3}$, we look at pentagons in $\mathcal{T}(S_{1,3})$ and hexagons in $\mathcal{C}_s(S_{1,3})$, which are cycles of minimal length. The case of $S_{1,p}$ with $p \geq 4$ is proved by inductive argument on p.

Remark 1.3. Let us describe some known facts on surfaces which are not dealt with in the above theorems. If $S = S_{0,p}$ is a surface of genus zero with $p \geq 5$, then both $\mathcal{T}(S)$ and $\mathcal{C}_s(S)$ are equal to $\mathcal{C}(S)$, and both $\mathcal{I}(S)$ and $\mathcal{K}(S)$ are equal to the pure mapping class group $\mathrm{PMod}(S)$ of S. The same conclusions as Theorems 1.1 and 1.2 thus hold.

The Birman exact sequence shows that $\mathcal{I}(S_{1,2})$ is isomorphic to $\pi_1(S_{1,1})$ and $\mathcal{K}(S_{1,2})$ is isomorphic to the commutator subgroup $[\pi_1(S_{1,1}), \pi_1(S_{1,1})]$. Mess [27] proved that $\mathcal{I}(S_{2,0}) = \mathcal{K}(S_{2,0})$ is isomorphic to the free group of infinite rank. It is easy to see that both $\mathcal{T}(S)$ and $\mathcal{C}_s(S)$ are zero-dimensional if $S = S_{1,2}$ or $S_{2,0}$. This shows that the same conclusions as Theorems 1.1 and 1.2 is not true for $S_{1,2}$ and $S_{2,0}$.

It is interesting to ask the same question for $S_{2,1}$. We know that the natural homomorphism from $\mathrm{Mod}^*(S_{2,1})$ into $\mathrm{Aut}(\mathcal{C}_s(S_{2,1}))$ is not surjective. Indeed, $\mathcal{C}_s(S_{2,1})$ consists of infinitely many ∞-regular trees. This is readily verified by combining the following facts:

- Let $\pi\colon \mathcal{C}_s(S_{2,1}) \to \mathcal{C}_s(S_{2,0})$ be the natural simplicial map obtained by filling a disk in the boundary of $S_{2,1}$. The fiber of π on each vertex of $\mathcal{C}_s(S_{2,0})$ is then a connected simplicial tree (see Section 7 of [21]).
- $\mathcal{C}_s(S_{2,0})$ is a zero-dimensional simplicial complex consisting of infinitely many vertices.

On the other hand, $\mathcal{T}(S_{2,1})$ is a connected graph, which contains a hexagon because one can embed $\mathcal{C}_s(S_{1,3})$ into $\mathcal{T}(S_{2,1})$ by gluing any two boundary components of $S_{1,3}$ (see Figure 7 for a hexagon in $\mathcal{C}_s(S_{1,3})$).

More generally, we study superinjective maps from the Torelli complex $\mathcal{T}(S)$ into itself when S is a surface of genus one. Superinjectivity of simplicial maps of $\mathcal{C}(S)$ was introduced by Irmak [12], [13] to study injective homomorphisms from finite index subgroups of $\mathrm{Mod}^*(S)$ into $\mathrm{Mod}^*(S)$ (see [1], [2] and [32] for related works). Superinjectivity of simplicial maps of $\mathcal{T}(S)$ is also defined similarly (see Section 2.2). As a result, we obtain the following:

Theorem 1.4. *Let* $S = S_{1,p}$ *be a surface of genus one with* $p \geq 3$. *Then the following assertions hold:*

   (i) *Any superinjective map* $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ *is an isomorphism.*
   (ii) *Let* Γ *be a finite index subgroup of* $\mathcal{I}(S)$ *and* $f\colon \Gamma \to \mathcal{I}(S)$ *an injective homomorphism. Then there exists a unique* $\gamma_0 \in \mathrm{Mod}^*(S)$ *satisfying the equation* $f(\gamma) = \gamma_0 \gamma \gamma_0^{-1}$ *for any* $\gamma \in \Gamma$. *In particular,* Γ *is co-Hopfian.*

Recall that a group Γ is said to be *co-Hopfian* if any injective homomorphism from Γ into itself is an isomorphism.

This paper is organized as follows. Section 2 collects the definition of simplicial complexes and subgroups of $\mathrm{Mod}^*(S)$ mentioned in this section. Their fundamental properties are also reviewed. Section 3 studies the Torelli complex $\mathcal{T}(S)$. A simplex of maximal dimension in $\mathcal{T}(S)$ is described. This is the first observation to prove that any superinjective map $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ preserves topological types of vertices of $\mathcal{T}(S)$. Section 4 presents a construction of a simplicial map $\Phi\colon \mathcal{C}(S) \to \mathcal{C}(S)$ which induces ϕ when the genus of S is equal to one. By using known facts on simplicial maps of $\mathcal{C}(S)$, we conclude that Φ is an automorphism of $\mathcal{C}(S)$ and obtain Theorem 1.1 (i) and Theorem 1.4 (i) for surfaces of genus one. In Section 5, given an automorphism ϕ of $\mathcal{C}_s(S)$, we construct an automorphism Φ of $\mathcal{C}(S)$ extending ϕ. The argument for surfaces of genus at least two heavily relies on [8]. Section 6 gives an algebraic characterization of twisting elements of the Torelli group $\mathcal{I}(S)$. This is used to associate an automorphism of $\mathcal{T}(S)$ (resp. $\mathcal{C}_s(S)$) to each element of $\mathrm{Comm}(\mathcal{I}(S))$ (resp. $\mathrm{Comm}(\mathcal{K}(S))$). The proof in Section 6 follows Vautaw's argument [33], [34] for closed surfaces. As a consequence, we compute $\mathrm{Comm}(\mathcal{I}(S))$ and $\mathrm{Comm}(\mathcal{K}(S))$. Section 7 describes the commensurator of the braid groups on the torus by using Theorem 1.1 for surfaces of genus one and the argument in [23]. In Appendix, we prove that each element of $\mathcal{I}(S)$ is pure in the sense of Ivanov [15]. This fact is used in Section 6.

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2. Complexes and groups associated with a surface

2.1. **Terminology.** Unless otherwise stated, we always assume that a surface is connected, compact and orientable, and it may have non-empty boundary. Let $S = S_{g,p}$ be a surface of genus g with p boundary components. The Euler characteristic of S is denoted by $\chi(S)$ and is equal to $-2g - p + 2$. A simple closed curve in S is said to be *essential* if it is neither homotopic to a single point of S nor isotopic to a boundary component of S.

When there is no confusion, we mean by a curve either an essential simple closed curve on S or an isotopy class of it. A curve a is said to be *separating* in S if $S \setminus a$ is disconnected, and otherwise a is said to be *non-separating* in S. A pair of

non-separating curves in S, $\{a, b\}$, is called a *bounding pair (BP)* in S if a and b are disjoint and not isotopic to each other and if $S \setminus (a \cup b)$ is disconnected. When we take into account an order of two curves of a bounding pair $\{a, b\}$, it is denoted by (a, b) and is called an *ordered bounding pair*. We often confuse a BP with and without an order if they can be distinguished in the context. We say that two non-separating curves in S are *BP-equivalent* in S if they are disjoint and form a BP in S.

Let a be a separating curve in S. If a cuts off a handle from S, then a is called an *h-curve* in S, where we mean by a *handle* a surface homeomorphic to $S_{1,1}$. If a cuts off a pair of pants from S, then a is called a *p-curve* in S, where we mean by a *pair of pants* a surface homeomorphic to $S_{0,3}$. A curve which is either an h-curve or a p-curve in S is called an *hp-curve* in S. If a BP $\{b, c\}$ in S cuts off a pair of pants from S, then $\{b, c\}$ is called a *p-BP* in S.

2.2. **The complex of curves and its variants.** We shall collect the definition of several abstract simplicial complexes arising from topology of simple closed curves on surfaces. The first complex was introduced by Harvey [11]. The second and third complexes (with an additional structure and for closed surfaces) were introduced by Farb and Ivanov [9].

The complex of curves. Let $V = V(S)$ denote the set of isotopy classes of essential simple closed curves on S and $\Sigma = \Sigma(S)$ denote the set of non-empty finite subsets σ of V such that all curves of σ can be realized disjointly on S at the same time. The *complex of curves*, denoted by $\mathcal{C} = \mathcal{C}(S)$, is the abstract simplicial complex such that the sets of vertices and simplices of $\mathcal{C}$ are given by V and Σ, respectively.

Let $i\colon V \times V \to \mathbb{N}$ denote the *geometric intersection number*, i.e., the minimal cardinality of the intersection of representatives for two elements of V. Given two simplices $\sigma = \{a_1, \ldots, a_n\}$ and $\tau = \{b_1, \ldots, b_m\}$ of $\mathcal{C}$, we define $i(\sigma, \tau)$ to be the sum $\sum_{k,l} i(a_k, b_l)$. We say that σ and τ are *disjoint* if $i(\sigma, \tau) = 0$, and otherwise we say that they *intersect*.

The complex of separating curves. The full subcomplex of $\mathcal{C}(S)$ spanned by all vertices corresponding to isotopy classes of separating curves is called the *complex of separating curves* and is denoted by $\mathcal{C}_s = \mathcal{C}_s(S)$. We denote the set of vertices and simplices of $\mathcal{C}_s$ by $V_s = V_s(S)$ and $\Sigma_s = \Sigma_s(S)$, respectively.

The Torelli complex. Let $V_{bp} = V_{bp}(S)$ be the set of isotopy classes of BPs in S. The *Torelli complex* for S, denoted by $\mathcal{T} = \mathcal{T}(S)$, is the abstract simplicial complex such that the set of vertices, denoted by $V_t = V_t(S)$, is given by the disjoint union $V_s \sqcup V_{bp}$, and a non-empty finite subset σ of $V_s \sqcup V_{bp}$ forms a simplex of $\mathcal{T}$ if and only if any two elements of σ are disjoint as elements of Σ. Let $\Sigma_t = \Sigma_t(S)$ denote the set of simplices of $\mathcal{T}$. We often regard an element of V_{bp} as an edge of $\mathcal{C}(S)$.

We note that if S is a surface of genus zero, then both $\mathcal{C}_s(S)$ and $\mathcal{T}(S)$ are equal to $\mathcal{C}(S)$ since any essential simple closed curve in S is separating in S.

Let us collect here terminology and symbols used throughout this paper. Let $\sigma \in \Sigma(S)$ be a simplex. A *BP-equivalence class* in σ is an equivalence class in the set of all non-separating curves of σ with respect to the BP-equivalence relation. When all curves of σ are non-separating and BP-equivalent to each other, we say that σ forms a BP-equivalence class. Two elements b_1, b_2 of V_{bp} are said to be

BP-equivalent if b_1 and b_2 are disjoint and the set of all curves in b_1 and b_2 forms a BP-equivalence class. An element of V_{bp} is called a *BP-vertex*. Similarly, an element of V corresponding to an h-curve and a p-curve is called an *h-vertex* and a *p-vertex*, respectively.

Let X be one of the simplicial complexes $\mathcal{C}(S)$, $\mathcal{C}_s(S)$ and $\mathcal{T}(S)$. We denote by $V(X)$ the set of vertices of X. Note that a map $\phi\colon V(X) \to V(X)$ defines a simplicial map from X into itself if and only if $i(\phi(a), \phi(b)) = 0$ for any two vertices $a, b \in V(X)$ with $i(a, b) = 0$. We mean by a *superinjective map* $\phi\colon X \to X$ a simplicial map $\phi\colon X \to X$ satisfying $i(\phi(a), \phi(b)) \neq 0$ for any two vertices $a, b \in V(X)$ with $i(a, b) \neq 0$. It is easy to see that any superinjective map $\phi\colon X \to X$ is injective.

Finally, given a surface S and a simplex $\sigma \in \Sigma(S)$, we denote by S_σ the surface obtained by cutting S along all curves in σ. When σ consists of a single curve a, we denote it by S_a for simplicity. If Q is a component of S_σ, then we have the natural inclusion of $V(Q)$ into $V(S)$.

2.3. **The mapping class group and its subgroups.** Let S be a surface. The *extended mapping class group* $\mathrm{Mod}^*(S)$ for S is the group consisting of all isotopy classes of homeomorphisms on S, where isotopy may move points in the boundary of S. The *mapping class group* $\mathrm{Mod}(S)$ for S is the subgroup of $\mathrm{Mod}^*(S)$ of index two consisting of all isotopy classes of orientation-preserving homeomorphisms on S. The *pure mapping class group* $\mathrm{PMod}(S)$ for S is the group consisting of all isotopy classes of orientation-preserving homeomorphisms on S which fix each boundary component of S as a set. The reader should consult [10] and [17] for fundamentals of these groups.

Given the isotopy class a of an essential simple closed curve in S, we denote by $t_a \in \mathrm{Mod}^*(S)$ the *(left) Dehn twist* about a. For an ordered BP $x = (a, b)$, we write $t_x = t_a t_b^{-1}$ and call it the *BP twist* about x. The *Torelli group* $\mathcal{I}(S)$ for S is the subgroup of $\mathrm{PMod}(S)$ generated by Dehn twists about all separating curves and BP twists about all BPs in S. The *Johnson kernel* $\mathcal{K}(S)$ for S is the subgroup of $\mathrm{PMod}(S)$ generated by Dehn twists about all separating curves in S. It is clear that both $\mathcal{I}(S)$ and $\mathcal{K}(S)$ are normal subgroups of $\mathrm{Mod}^*(S)$. We refer to [30] for variants of the definition of the Torelli group. Note that if S is a surface of genus zero, then any curve in S is separating, and thus both $\mathcal{I}(S)$ and $\mathcal{K}(S)$ are equal to $\mathrm{PMod}(S)$.

Let $S = S_{g,p}$ be a surface. It is known that if $g \geq 2$ and $p = 0, 1$, then $\mathcal{I}(S)$ is equal to the subgroup of $\mathrm{Mod}(S)$ consisting of all elements which trivially act on the homology group $H_1(S, \mathbb{Z})$. This is due to Powell [29] (for closed surfaces and based on Birman's work [6] on $Sp(2g, \mathbb{Z})$) and Johnson [18]. This description is the original definition of the Torelli group for S with $p = 0, 1$. Afterward, Johnson [20] produced a finite generating set for $\mathcal{I}(S)$ consisting of BP twists when $g \geq 3$ and $p = 0, 1$. In contrast, if $g = 2$ and $p = 0$, then $\mathcal{I}(S)$ is not finitely generated. Indeed, $\mathcal{I}(S)$ is isomorphic to the free group of infinite rank (see [26], [27] and [4]). The following fact on $\mathcal{K}(S)$ is fundamental and will be used in Section 6.

Theorem 2.1 ([19])**.** *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \leq 1$. Then $\mathcal{K}(S)$ contains no non-zero power of a BP twist.*

Proposition 2.2. *If $S = S_{g,p}$ is a surface with either* (a) *$g \geq 2$ and $p \geq 0$; or* (b) *$g = 1$ and $p \geq 2$, then $\mathcal{K}(S)$ contains no non-zero power of a BP twist.*

Proof. We first assume the condition (a) and $p \geq 2$. Let $x = (a, b)$ be an ordered BP in S. If x does not cut off a surface of genus zero, then fill disks in any $p - 1$ components of ∂S. Otherwise, choose one component of ∂S contained in the surface of genus zero cut off by x, and fill disks in all of the other components of ∂S. One then obtains the surface Q homeomorphic to $S_{g,1}$ and the homomorphism $q \colon \mathrm{PMod}(S) \to \mathrm{Mod}(Q)$. Note that x can be seen as an ordered BP in Q. It follows from $q(\mathcal{K}(S)) = \mathcal{K}(Q)$ and Theorem 2.1 that no non-zero power of t_x is contained in $\mathcal{K}(S)$.

We next assume the condition (b). Once the conclusion for $S = S_{1,2}$ is shown, the conclusion for the other cases is verified along the argument in the previous paragraph. Put $S = S_{1,2}$ and let R be the surface obtained by filling a disk in one boundary component of S, which is homeomorphic to $S_{1,1}$. One then obtains the Birman exact sequence

$$1 \to \pi_1(R) \xrightarrow{i} \mathrm{PMod}(S) \to \mathrm{Mod}(R) \to 1,$$

and $\pi_1(R)$ is the free group generated by two standard non-separating simple loops a, b in R. It follows from the definition of i that both $i(a)$ and $i(b)$ are BP twists in $\mathrm{PMod}(S)$. Thus, $i(\pi_1(R)) < \mathcal{I}(S)$. Note that $i([a, b])$ is the Dehn twist about a separating curve in S, where $[a, b]$ denotes the commutator of a and b. Since the actions of $\mathrm{PMod}(S)$ on the set of all BPs and the set of all separating curves in S are both transitive, the normal closure of $i(a)$ in $\mathrm{PMod}(S)$ is equal to $\mathcal{I}(S)$. Therefore, $i(\pi_1(R)) = \mathcal{I}(S)$. The same argument proves that the image of the commutator subgroup $[\pi_1(R), \pi_1(R)]$ via i is equal to $\mathcal{K}(S)$. It is clear that no non-zero power of $i(a)$ and its conjugate in $\mathrm{PMod}(S)$ is contained in $i([\pi_1(R), \pi_1(R)])$. $\qquad \square$

Since $\mathrm{Mod}^*(S)$ naturally acts on the complex of curves, $\mathcal{C}(S)$, we have the natural homomorphism π from $\mathrm{Mod}^*(S)$ into the simplicial automorphism group $\mathrm{Aut}(\mathcal{C}(S))$ of $\mathcal{C}(S)$. The following theorem is a fundamental tool to compute commensurators of mapping class groups and their subgroups. It is proved in [16], [22] and [24].

Theorem 2.3. *Let $S = S_{g,p}$ be a surface with $3g + p - 4 > 0$, and let $\pi \colon \mathrm{Mod}^*(S) \to \mathrm{Aut}(\mathcal{C}(S))$ be the natural homomorphism. Then*

  (i) *if $(g, p) \neq (1, 2), (2, 0)$, then π is an isomorphism.*
  (ii) *if $(g, p) = (1, 2)$, then $\ker \pi$ is equal to the center of $\mathrm{Mod}^*(S)$, and the image of π is equal to the group of automorphisms of $\mathcal{C}(S)$ preserving vertices corresponding to separating curves, which is a finite index subgroup of $\mathrm{Aut}(\mathcal{C}(S))$.*
  (iii) *if $(g, p) = (2, 0)$, then $\ker \pi$ is equal to the center of $\mathrm{Mod}^*(S)$, and π is surjective.*

It is known that if $(g, p) = (1, 2), (2, 0)$, then the center of $\mathrm{Mod}^*(S)$ consists of exactly two elements. Any superinjective map from $\mathcal{C}(S)$ into itself is shown to be an isomorphism in [1], [2], [12] and [13]. More generally, the following theorem is obtained.

Theorem 2.4 ([32]). *Let $S = S_{g,p}$ be a surface with $3g + p - 4 > 0$. Then any injective simplicial map $\phi \colon \mathcal{C}(S) \to \mathcal{C}(S)$ is an isomorphism.*

3. Basics of the Torelli complex

This section proves that any superinjective map of the Torelli complex and the complex of separating curves preserves the topological type of each vertex.

3.1. **Simplices of maximal dimension.** A description of simplices of maximal dimension in the Torelli complex is given. The next observation on BPs will be used many times in what follows.

Lemma 3.1. *Let a be a BP in S, and let b be either a separating curve in S disjoint from a or a BP in S which is disjoint from a and is not BP-equivalent to a. Then the two curves in a are contained in a single component of S_b.*

Proof. Let a_1 and a_2 denote the non-separating curves in a. Assume b to be a separating curve. One can easily check that each a_j is non-separating in the component of S_b containing it. If a_1 and a_2 were contained in different components of S_b, then $S \setminus (a_1 \cup a_2)$ would be connected. This is a contradiction. We next assume b is a BP. Note that each separating curve in a component of S_b is either separating in S or forms a BP in S with any curve of b. Since a and b are not BP-equivalent, each curve in a is non-separating in the component of S_b containing it. The conclusion of the lemma then follows as before. $\square$

Lemma 3.2. *Let S be a surface and let $b, c \in \Sigma(S)$ be simplices such that*
- *$|b| \geq 2$, $|c| \geq 2$, $b \cap c = \emptyset$ and $b \cup c \in \Sigma(S)$; and*
- *each curve of b and c is non-separating, and each of b and c forms a BP-equivalence class in the simplex $b \cup c$.*

Then the following assertions hold:
- (i) *For each component Q of S_b, there exist exactly two curves of b corresponding to boundary components of Q.*
- (ii) *There exists a unique component R of S_b such that $c \in \Sigma(R)$. Moreover, each curve of c is non-separating in R, and c forms a BP-equivalence class as an element of $\Sigma(R)$.*

Proof. If Q is a component of S_b, then the number of curves of b corresponding to boundary components of Q is at least two because each curve of b is non-separating in S. Assume that the number is at least three. Choose three curves b_1, b_2 and b_3 of b corresponding to boundary components of Q, and take a p-curve a in Q cutting off a pair of pants whose boundary contains b_1 and b_2. Since the pair $\{b_1, b_2\}$ separates S into two components, a is separating in S. This contradicts Lemma 3.1 because both components of S_a contain a curve of b. This proves the assertion (i).

Take a curve $c_1 \in c$ and a component R of S_b with $c_1 \in V(R)$. We first claim that c_1 is non-separating in R. Assume otherwise, and let $b_1, b_2 \in b$ be curves corresponding to boundary components of R. If both b_1 and b_2 were contained in a single component of R_{c_1}, then c_1 would be separating in S. This is a contradiction. Otherwise, b_1 and c_1 form a BP in S, and this is also a contradiction. Therefore, c_1 is non-separating in R.

Suppose that a curve $c_2 \in c$ different from c_1 is contained in a component R' of S_b different from R. The curve c_2 is then non-separating in R', and $S \setminus (c_1 \cup c_2)$ is connected because any two curves of b can be connected by an arc in S which does not intersect c_1 and c_2. Since $\{c_1, c_2\}$ is a BP in S, this is a contradiction and proves that each curve of c is contained in R. It is clear that c forms a BP-equivalence class as an element of $\Sigma(R)$. $\square$

Lemma 3.3. *Let $S = S_{g,p}$ be a surface and let σ be a simplex of $\mathcal{C}(S)$ consisting of separating curves in S. Then the following assertions hold:*

(i) *The inequality $|\sigma| \leq 2g + p - 3$ holds and this equality can be attained. In particular, $\dim(\mathcal{C}_s(S)) = 2g + p - 4$.*
(ii) *If $|\sigma| = 2g + p - 3$, then S_σ consists of g handles and $g + p - 2$ pairs of pants.*

Proof. If there exists a component Q of S_σ which is neither a pair of pants nor a handle, then any separating curve in Q is separating in S. This implies that once the assertion (i) is proved, the assertion (ii) follows. To prove the assertion (i), we may assume that each component of S_σ is either a pair of pants or a handle. Note that the number of components of S_σ is equal to $|\chi(S)| = 2g + p - 2$. The assertion (i) is then verified by the counting argument of the number of boundary components of components of S_σ. $\square$

Proposition 3.4. *Let $S = S_{g,p}$ be a surface with $g \geq 1$ and $g + p \geq 3$. Then*

$$\dim(\mathcal{T}(S)) = (g - 1) + \binom{g + p - 1}{2} - 1.$$

Moreover, if $g + p \geq 4$, then for any simplex σ of $\mathcal{T}(S)$ of maximal dimension, there exists a unique simplex $\{\alpha_1, \ldots, \alpha_{g-1}, \beta_1, \ldots, \beta_{g+p-1}\}$ of $\mathcal{C}(S)$ such that

(a) *each of $\alpha_1, \ldots, \alpha_{g-1}$ is an h-curve;*
(b) *each of $\beta_1, \ldots, \beta_{g+p-1}$ is non-separating, and the family $\{\beta_1, \ldots, \beta_{g+p-1}\}$ forms a BP-equivalence class; and*
(c) *σ consists of $\alpha_1, \ldots, \alpha_{g-1}$ and all BPs of two curves in $\{\beta_1, \ldots, \beta_{g+p-1}\}$.*

Proof. It is easy to find a simplex $\{\alpha_1, \ldots, \alpha_{g-1}, \beta_1, \ldots, \beta_{g+p-1}\}$ of $\mathcal{C}(S)$ satisfying the conditions (a) and (b). One can verify that the dimension of the simplex of $\mathcal{T}(S)$ containing all α_i's and all BPs of two curves chosen from the family $\{\beta_1, \ldots, \beta_{g+p-1}\}$ is equal to the right hand side of the equality in the proposition. Therefore, $\dim(\mathcal{T}(S))$ is not smaller than this number.

We prove the equation in the proposition by induction on $g + p$. When $g + p = 3$, we have $(g, p) = (1, 2), (2, 1), (3, 0)$, and it is then easy to see $\dim(\mathcal{T}(S)) = g - 1$. Note that the cardinality of BP-vertices in a simplex of $\mathcal{T}(S)$ is at most one.

In what follows, we assume $g + p \geq 4$. Let $\sigma \in \Sigma_t(S)$ be a simplex of maximal dimension. Let

$$s = \{a_1, \ldots, a_k, b_{11}, \ldots, b_{1m_1}, b_{21}, \ldots, b_{lm_l}\}$$

be the collection of curves of elements in σ such that

- each of $a_1, \ldots, a_k$ is separating in S; and
- b_{ij} is non-separating in S for each i, j, and the family $b_i = \{b_{i1}, \ldots, b_{im_i}\}$ forms a BP-equivalence class in s for each i.

Since the dimension of σ is maximal, σ contains the BP of any two curves in b_i for any i, and we have

$$|\sigma| = k + \sum_{i=1}^{l} \binom{m_i}{2}$$

The following two claims show that s contains exactly one BP-equivalence class.

Claim 3.5. *We have $l \leq 1$.*

Proof. We suppose $l > 1$ and deduce a contradiction. By Lemma 3.2, there exists a unique component R of S_{b_1} containing all curves of b_2. After exchanging the indices, we may assume that

- b_{11} and b_{12} are curves of b_1 corresponding to boundary components of R; and
- the curves $a_1, \ldots, a_{k'}$ and all curves in $b_2, \ldots, b_{l'}$ with $0 \leq k' \leq k$ and $2 \leq l' \leq l$ form the family of all curves in s contained in R. Note that the curves $a_1, \ldots, a_{k'}$ and all BPs obtained from $b_2, \ldots, b_{l'}$ form the simplex $\sigma_R = \sigma \cap V_t(R)$ of $\mathcal{T}(R)$.

Since b_{11} and b_{12} are contained in a single component of the surface obtained by cutting R along each element of σ_R, there exists a p-curve c in R such that $i(\sigma_R, c) = 0$ and c cuts off a pair of pants whose boundary contains b_{11} and b_{12}. Note that c is separating in S and has to be contained in σ_R since $\dim \sigma$ is maximal. It follows that c is equal to one of $a_1, \ldots, a_{k'}$. Let R' be the component of R_c which does not contain b_{11} and b_{12}. The simplex $\sigma_R \setminus \{c\} \in \Sigma_t(R)$ can be seen as an element of $\Sigma_t(R')$.

Let g_1 be the genus of R and p_1 the number of boundary components of R. Note that g_1 is positive because b_2 is contained in R. Since R' is a surface of genus g_1 with $p_1 - 1$ boundary components, the hypothesis of the induction implies the inequality

$$|\sigma_R \setminus \{c\}| = (k' - 1) + \sum_{i=2}^{l'} \binom{m_i}{2} \leq (g_1 - 1) + \binom{g_1 + p_1 - 2}{2}.$$

Choose a simplex $\{c_1, \ldots, c_{g_1+p_1-3}\} \in \Sigma(R)$ such that for each i, each c_i is separating in R, and b_{11} and b_{12} are not contained in a single component of R_{c_i}. One can then find a simplex $\{d_1, \ldots, d_{g_1}\} \in \Sigma(R)$ consisting of h-curves disjoint from all c_i's. After deleting curves in σ_R from s, add all c_i's and d_j's to it. This new collection of curves is denoted by s_1 and associates the simplex $\sigma_1 \in \Sigma_t(S)$ containing all separating curves in s_1 and all BPs of curves in s_1. We then obtain the equality

$$|\sigma_1| = k - k' + g_1 + \binom{m_1 + g_1 + p_1 - 3}{2} + \sum_{i=l'+1}^{l} \binom{m_i}{2}$$

and the inequality

$$|\sigma_1| - |\sigma| \geq (m_1 - 1)(g_1 + p_1 - 3)$$

by using the inequality shown above. If $g_1 + p_1 = 3$, then b_2 cannot be in R. Therefore, the right hand side in the last inequality is positive. This contradicts the maximality of $\dim \sigma$. □

Claim 3.6. *We have $l > 0$.*

Proof. If $l = 0$, then σ consists of separating curves in S. Lemma 3.3 implies that $|\sigma| \leq 2g + p - 3$. The inequality

$$(g - 1) + \binom{g + p - 1}{2} - (2g + p - 3) = \frac{(g + p - 2)(g + p - 3)}{2} > 0$$

then holds when $g + p \geq 4$. This is a contradiction. □

These two claims show that s is of the form

$$s = \{a_1, \ldots, a_k, b_1, \ldots, b_m\},$$

where each a_i is separating in S and any two curves in the family $b = \{b_1, \ldots, b_m\}$ are BP-equivalent in S. We put $T = S_{b_m}$ and $r = s \setminus \{b_m\}$. Any curve in r is then separating in T. It follows from maximality of $\dim \sigma$ that the surface T_r consists of

$g - 1$ handles and $g + p - 1$ pairs of pants. Applying Lemma 3.3 to T, we obtain the equation $k + (m - 1) = 2(g - 1) + (p + 2) - 3$. Since any b_j cannot be the boundary of a handle in T_r, we have the inequality $k \geq g - 1$. Thus, $m \leq g + p - 1$. The inequality $g + p - 4 \geq 0$ then implies the inequality

$$|\sigma| = k + \binom{m}{2} = 2g + p - 2 + \frac{m(m - 3)}{2}$$

$$\leq 2g + p - 2 + \frac{(g + p - 1)(g + p - 4)}{2} = g - 1 + \binom{g + p - 1}{2}.$$

This proves the equality in the proposition. The equality in the last inequality holds if and only if $k = g - 1$ and $m = g + p - 1$. The above argument shows that each a_i is an h-curve. Therefore, we proved that s satisfies the conditions (a), (b) and (c). The uniqueness of s is obvious. $\square$

Lemma 3.7. *Let $S = S_{g,p}$ be a surface with $g \geq 1$. Suppose either $g + p \geq 4$ or $(g, p) = (3, 0)$. Let $\phi \colon \mathcal{T}(S) \to \mathcal{T}(S)$ be a superinjective map. Then the following assertions hold:*

- (i) *If $g \geq 2$, then the inclusions $\phi(V_{bp}) \subset V_{bp}$ and $\phi(V_s) \subset V_s$ hold.*
- (ii) *If $g = 1$, then the inclusion $\phi(V_{bp}) \subset V_{bp}$ holds.*
- (iii) *If b_1 and b_2 are disjoint BPs in S and are BP-equivalent, then $\phi(b_1)$ and $\phi(b_2)$ are also BP-equivalent.*

Proof. The assertion (ii) is a direct consequence of Proposition 3.4. Assume $g \geq 2$ and $g + p \geq 4$. The same proposition implies that if $a \in V$ is an h-vertex, then $\phi(a)$ is either an h-vertex or a BP-vertex. Note that for any simplex $\sigma \in \Sigma_t(S)$ of maximal dimension, a vertex a in σ is an h-vertex if and only if there exists a vertex $b \in V_t$ with $i(a, b) \neq 0$ and $i(c, b) = 0$ for each vertex $c \in \sigma \setminus \{a\}$. Therefore, if $a \in V$ is an h-vertex, then so is $\phi(a)$. This also implies the inclusion $\phi(V_{bp}) \subset V_{bp}$ and the assertion (iii) by Proposition 3.4.

Let $b \in V_s$ be a vertex which is not an h-vertex. Choose h-vertices $b_1, \ldots, b_g \in V$ such that $\{b, b_1, \ldots, b_g\}$ is a g-simplex of $\mathcal{C}_s(S)$. If $\phi(b) \in V_{bp}$, then this contradicts the fact that all of $\phi(b_1), \ldots, \phi(b_g)$ are h-vertices. Thus, $\phi(b) \in V_s$. This shows the inclusion $\phi(V_s) \subset V_s$.

When $(g, p) = (3, 0)$, any simplex σ of $\mathcal{T}(S)$ of maximal dimension consists of either three h-vertices or two h-vertices and one BP-vertex. Note that the former occurs if and only if for each vertex $\alpha \in \sigma$, there exists a vertex $\beta \in V_t$ satisfying $i(\alpha, \beta) = 0$ and $i(\gamma, \beta) \neq 0$ for each vertex $\gamma \in \sigma \setminus \{\alpha\}$. This implies that ϕ preserves h-vertices. If we have a BP-vertex $b \in V_{bp}$ with $\phi(b)$ an h-vertex, then choose four distinct h-curves c_1, c_2, c_3 and c_4 disjoint from b such that c_1 and c_2 are contained in a single component of S_b and c_3 and c_4 are contained in another component of S_b. It then follows that $\phi(c_1)$, $\phi(c_2)$, $\phi(c_3)$ and $\phi(c_4)$ are h-curves in the component of $S_{\phi(b)}$, denoted by Q, that is not a handle and is homeomorphic to $S_{2,1}$. On the other hand, the two subsurfaces filled by $\phi(c_1)$ and $\phi(c_2)$ and by $\phi(c_3)$ and $\phi(c_4)$ are disjoint. This contradicts $|\chi(Q)| = 3$. $\square$

3.2. **The case $g = 1$ and $p = 3$.** We put $S = S_{1,3}$ throughout this subsection. We say that a 5-tuple $(v_1, \ldots, v_5)$ of vertices of $\mathcal{T}(S)$ forms a *pentagon* in $\mathcal{T}(S)$ if $i(v_j, v_{j+1}) = 0$ and $i(v_j, v_{j+2}) \neq 0$ for each j mod 5 (see Figure 1).

Lemma 3.8. *There exists no pentagon in $\mathcal{T}(S)$ consisting of only BP-vertices.*
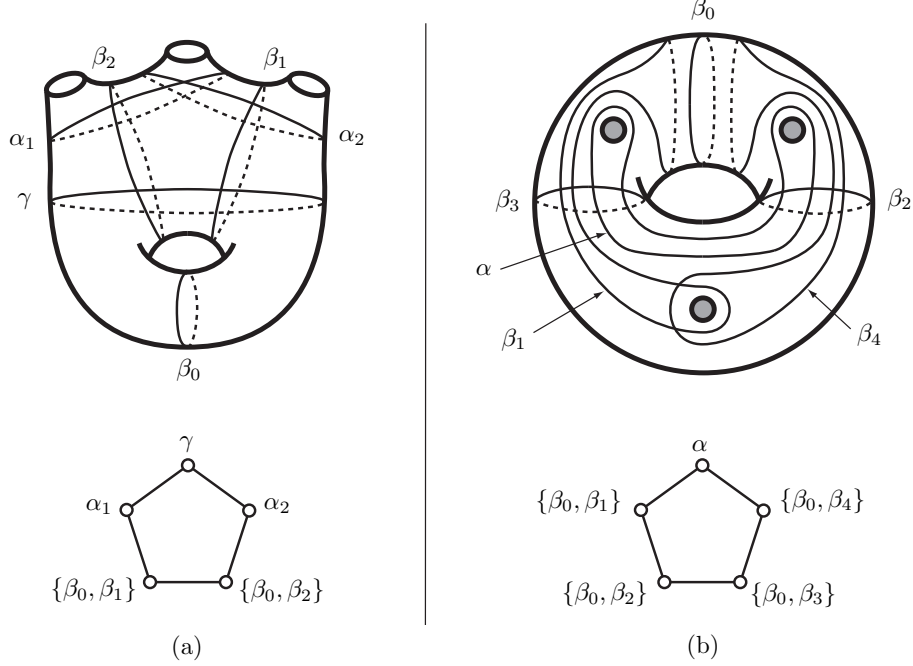
FIGURE 1. Two pentagons in $\mathcal{T}(S_{1,3})$

Proof. We assume that there exists a 5-tuple $(v_1, \ldots, v_5)$ forming a pentagon in $\mathcal{T}(S)$ with $v_i \in V_{bp}$ for each i. We denote by ∂_1, ∂_2 and ∂_3 the three components of ∂S. For each BP b in S, one can associate the index $j \in \{1, 2, 3\}$ such that ∂_j is contained in the pair of pants cut off by b. Let us denote this association by $\theta \colon V_{bp} \to \{1, 2, 3\}$. Since different indices are associated to two adjacent vertices in V_{bp}, we may assume

$$\theta(v_1) = 3, \quad \theta(v_2) = \theta(v_4) = 1, \quad \theta(v_3) = \theta(v_5) = 2.$$

Let R be the surface obtained by filling a disk in ∂_3. We denote by $\mathcal{C}^*(R)$ the simplicial cone over $\mathcal{C}(R)$ and denote by $*$ the cone point. One can define the natural simplicial map $\pi \colon \mathcal{C}(S) \to \mathcal{C}^*(R)$ so that $\pi^{-1}(\{*\})$ consists of all p-curves in S cutting off a pair of pants containing ∂_3. It is easy to check the following facts:

- If $u, v \in V_{bp}(S)$ satisfy $i(u, v) = 0$, $\theta(u) = 1$ and $\theta(v) = 2$, then $\pi(u) = \pi(v)$ and it is an edge of $\mathcal{C}(R)$.
- If $w \in V_{bp}(S)$ satisfies $\theta(w) = 3$, then $\pi(w)$ consists of exactly one vertex of $\mathcal{C}(R)$ corresponding to a non-separating curve in R.
- If $a, b \in V(S)$ and $w \in V_{bp}(S)$ satisfy $i(a, b) \neq 0$, $i(a, w) = i(b, w) = 0$ and $\theta(w) = 3$, then $\pi(a), \pi(b) \in V(R)$ and $i(\pi(a), \pi(b)) \neq 0$.

The first fact implies $\pi(v_2) = \pi(v_3) = \pi(v_4) = \pi(v_5)$. We put $v_1 = \{\alpha_0, \alpha_1\}$ and $v_2 = \{\alpha_0, \alpha_2\}$. Note that any two adjacent vertices in $V_{bp}(S)$ have a common curve. One of the two curve in v_5, say α, belongs to v_1, and the other curve β in v_5 intersects v_2 and is disjoint from v_1. Thus, β intersects α_2. The third fact implies $i(\pi(\alpha_2), \pi(\beta)) \neq 0$. This contradicts $\pi(v_2) = \pi(v_5)$. $\qquad\square$

Lemma 3.9. *Let $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ be a superinjective map, and let a be a p-vertex of $\mathcal{T}(S)$. Then $\phi(a)$ is also a p-vertex of $\mathcal{T}(S)$.*

Proof. The image of the pentagon in Figure 1 (b) via ϕ contains four BP-vertices by Lemma 3.7 (ii). Lemma 3.8 implies that the other vertex is not a BP-vertex. Since any separating curve in S which is disjoint from a BP is a p-curve, we obtain the lemma. □

Lemma 3.10. *Suppose that we are given a 5-tuple $(v_1, \dots, v_5)$ of vertices of $\mathcal{T}(S)$ forming a pentagon in $\mathcal{T}(S)$ such that both v_3 and v_4 are BP-vertices and both v_2 and v_5 are p-vertices. Then v_1 is an h-vertex.*

Proof. It is clear that v_1 is not a p-vertex because there is no p-vertex in the link of a p-vertex in $\mathcal{T}(S)$. We number components of ∂S from one to three as in the proof of Lemma 3.8. For each BP or p-curve a in S, let Q be the component of S_a containing only one component of ∂S. One then associates to a the number of the component of ∂S contained in Q. Note that different numbers are associated to two adjacent vertices in V_{bp} and that the same number is associated to a p-vertex and a BP-vertex which are adjacent. Therefore, we may assume that v_2 and v_3 are associated the number 1 and that v_4 and v_5 are associated the number 2. It follows that v_1 is not a BP-vertex, and thus it is an h-vertex. □

Lemmas 3.9 and 3.10 imply the following:

Proposition 3.11. *Put $S = S_{1,3}$ and let $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ be a superinjective map. Then ϕ preserves p-vertices and h-vertices in $\mathcal{T}(S)$, respectively. In particular, the inclusion $\phi(V_s) \subset V_s$ holds.*

3.3. **The case $g = 1$ and $p \geq 4$.** The aim of this subsection is to prove that any superinjective map from $\mathcal{T}(S)$ into itself preserves vertices corresponding to separating curves in the case of $S = S_{1,p}$ with $p \geq 4$. We first introduce rooted simplices of $\mathcal{T}(S)$ to prove it.

Definition 3.12. Let S be a surface, and let σ be a simplex of $\mathcal{T}(S)$ consisting of BP-vertices. We say that σ is *rooted* if σ forms a BP-equivalence class and there exists a non-separating curve α in S contained in any BPs of σ. If $|\sigma| \geq 2$, then α is uniquely determined and called the *root curve* for σ.

Lemma 3.13. *Let $S = S_{g,p}$ be a surface with $g \geq 1$, and suppose either $g + p \geq 4$ or $(g, p) = (3, 0)$. Let $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ be a superinjective map, and let σ be a simplex of $\mathcal{T}(S)$ consisting of BP-vertices. If σ is rooted, then so is $\phi(\sigma)$.*

Proof. We first prove that ϕ preserves rooted simplices consisting of $g + p - 2$ BP-vertices. Let $\sigma = \{b_1, \dots, b_n\}$ be such a simplex, where we put $n = g + p - 2$. It then follows that for each j, there exists a vertex $a_j \in V_t$ such that $i(a_j, b_j) \neq 0$ and $i(a_j, b_k) = 0$ for each $k \neq j$. This implies that for each j, there exists a non-separating curve c_j contained in $\phi(b_j)$, but not in $\phi(b_k)$ for each $k \neq j$. Let c_0 be the curve in $\phi(b_1)$ that is not equal to c_1. Note that $c_0, c_1, \dots, c_n$ are pairwise distinct curves and BP-equivalent to each other by Lemma 3.7 (iii). It follows from Proposition 3.4 that there exist at most $n + 1$ non-separating curves in S such that each BP of $\phi(\sigma)$ consists of two of them. We therefore obtain the equality $\phi(b_j) = \{c_0, c_j\}$ for each j. This shows $\phi(\sigma)$ is rooted.

Since a simplex of $\mathcal{T}(S)$ consisting of BP-vertices is rooted if and only if it is contained in a rooted simplex consisting of $g + p - 2$ BP-vertices, the lemma follows. $\square$

Lemma 3.14. *Let S be a surface in Lemma 3.13, and let $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ be a superinjective map. Suppose that we are given two distinct and disjoint BPs $b_1 = \{\alpha_0, \alpha_1\}$ and $b_2 = \{\alpha_0, \alpha_2\}$ in S with the common curve α_0. We put $\phi(b_i) = \{\beta_0, \beta_i\}$ for each $i = 1, 2$. Then we have $\phi(\{\alpha_1, \alpha_2\}) = \{\beta_1, \beta_2\}$.*

Proof. Note that Lemma 3.13 implies that $\phi(b_1)$ and $\phi(b_2)$ have the common curve β_0. If the conclusion of the lemma were not true, then $\phi(\{\alpha_1, \alpha_2\})$ would be a BP containing β_0 by Lemma 3.13. In general, the maximal dimension of rooted simplices in any simplex of maximal dimension in $\mathcal{T}(S)$ is equal to $g + p - 3$. By choosing a simplex of maximal dimension in $\mathcal{T}(S)$ containing b_1 and b_2 and by using injectivity of ϕ, one can then deduce a contradiction. $\square$

Lemma 3.15. *Let $S = S_{1,p}$ be a surface with $p \geq 4$, and let $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ be a superinjective map. If $a \in V_s$ is a p-curve, then $\phi(a) \in V_s$ and it is a p-curve.*

Proof. Let $a \in V_s$ be a p-curve, and choose a rooted simplex σ consisting of $p - 2$ BPs in S disjoint from a. Once it is shown that $\phi(a)$ belongs to V_s, one can easily verify that $\phi(a)$ is a p-curve because any separating curve in S disjoint from the rooted simplex $\phi(\sigma)$ is a p-curve.

We assume $\phi(a) \in V_{bp}$. It then follows that $\phi(a)$ and each BP in $\phi(\sigma)$ are BP-equivalent since the genus of S is equal to one. Note that $p - 1$ non-separating curves in S appear in BPs of $\phi(\sigma)$ and that at most p non-separating curves in S appear in BPs of any simplex of $\mathcal{T}(S)$. Therefore, there exists a BP b in σ such that $\phi(a)$ and $\phi(b)$ have a common curve.

Assume that this common curve is not equal to the root curve for $\phi(\sigma)$. Let b' be a BP of σ different from b. We then have $\phi(b) \subset \phi(a) \cup \phi(b')$. This contradicts the existence of a vertex $c \in V_t$ with $i(c, b) \neq 0$ and $i(c, a) = i(c, b') = 0$. Therefore, the common curve of $\phi(a)$ and $\phi(b)$ is the root curve for $\phi(\sigma)$.

Let $\alpha_0, \alpha_1, \ldots, \alpha_{p-2}$ be all non-separating curves in BPs of σ. By Lemma 3.14, ϕ induces the map ϕ_σ from the set of all curves of BPs in σ into the set of all curves of BPs in $\phi(\sigma)$ such that $\phi(\{\alpha_i, \alpha_j\}) = \{\phi_\sigma(\alpha_i), \phi_\sigma(\alpha_j)\}$ for any distinct i, j. Since for each i, the family $\{\{\alpha_i, \alpha_j\} \mid j \neq i\}$ is a rooted simplex disjoint from a, the argument of the previous paragraph shows that $\phi(a)$ contains $\phi_\sigma(\alpha_i)$ for each i. This is a contradiction. $\square$

Proposition 3.16. *Let $S = S_{1,p}$ be a surface with $p \geq 3$, and let $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ be a superinjective map. Then $\phi(V_s) \subset V_s$.*

Proof. By Lemma 3.15, it suffices to show that if $a \in V_s$ is a separating curve which is not a p-curve, then $\phi(a) \in V_s$. We prove this claim by induction on p. When $p = 3$, this is proved in Proposition 3.11. Assume $p \geq 4$ and let $a \in V_s$ be a curve which is not a p-curve. One can then find a p-curve $\alpha \in V_s$ disjoint from a. The map ϕ induces the superinjective map $\phi_\alpha\colon \mathcal{T}(R^\alpha) \to \mathcal{T}(R^{\phi(\alpha)})$, where for a p-curve β in S, we denote by R^β the component of S_β that is not a pair of pants. Since R^α and $R^{\phi(\alpha)}$ are homeomorphic and since the number of boundary components of R^α is less than the one of S, the hypothesis of the induction implies that $\phi_\alpha(V_s(R^\alpha)) \subset V_s(R^{\phi(\alpha)})$. It follows from $a \in V_s(R^\alpha)$ that $\phi(a) \in V_s(R^{\phi(\alpha)})$, and thus $\phi(a) \in V_s(S)$. $\square$

3.4. **Superinjective maps of the complex of separating curves.** We have proved that any superinjective map $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ preserves vertices corresponding to separating curves in S. It follows that ϕ induces a superinjective map from $\mathcal{C}_s(S)$ into $\mathcal{C}_s(S)$. This subsection studies general superinjective maps from $\mathcal{C}_s(S)$ into itself.

Let σ be a simplex of maximal dimension in $\mathcal{C}_s(S)$. We say that two curves a and b in σ are *adjacent with respect to σ* if there exists a component of S_σ containing a and b as boundary components. We define the *adjacency graph $\mathcal{G}(\sigma)$* for σ as the simplicial graph consisting of a vertex for each curve in σ and edges corresponding to adjacency with respect to σ. Adjacency graphs for simplices of maximal dimension in $\mathcal{C}(S)$ are introduced by Irmak [12] in the work on superinjective maps of $\mathcal{C}(S)$.

Lemma 3.17. *Let $S = S_{g,p}$ be a surface with $|\chi(S)| \geq 4$, and let $\phi\colon \mathcal{C}_s(S) \to \mathcal{C}_s(S)$ be a superinjective map. Then for any simplex σ of maximal dimension in $\mathcal{C}_s(S)$, ϕ induces an isomorphism between the adjacency graphs of σ and $\phi(\sigma)$.*

Proof. The proof follows Lemma 5.1 in [1]. It is enough to prove that ϕ preserves adjacency and non-adjacency with respect to σ. Note that two curves in σ are adjacent with respect to σ if and only if there exists a separating curve in S which intersect both of them and is disjoint from any other curve of σ. This proves that ϕ preserves adjacency.

Let $\sigma = \{a_1, b_1, c_1, \ldots, c_n\}$, where $n = 2g + p - 5$. If a_1 and b_1 are not adjacent with respect to σ, then one can find two separating curves a_2 and b_2 such that $\{a_i, b_j, c_1, \ldots, c_n\}$ is a simplex of maximal dimension in $\mathcal{C}_s(S)$ for any $i, j \in \{1, 2\}$. It is then obvious that a_1, b_1, a_2 and b_2 form a square in $\mathcal{C}_s(S)$ in this order.

Conversely, if a_1 and b_1 are adjacent with respect to σ, then there exists a component Q of S_c containing a_1 and b_1, where we put $c = \{c_1, \ldots, c_n\}$. One can easily show $|\chi(Q)| = 3$. Since there exists no square in $\mathcal{C}_s(Q)$, this proves that ϕ preserves non-adjacency. $\square$

Lemma 3.18. *Let $S = S_{g,p}$ be a surface with $|\chi(S)| \geq 2$ and let $\phi\colon \mathcal{C}_s(S) \to \mathcal{C}_s(S)$ be a superinjective map. Then the following assertions hold:*

- (i) *For each $a \in V_s$, let Q_1, Q_2 be components of S_a and let R_1, R_2 be components of $S_{\phi(a)}$. Then the inclusion $\phi(V_s(Q_i)) \subset V_s(R_i)$ holds for each i after exchanging the indices if necessary.*
- (ii) *ϕ is χ-preserving, i.e., $\chi(Q_i) = \chi(R_i)$ for each i in the assertion (i).*

Proof. The claim in the case $|\chi(S)| \leq 3$ is obvious. Assume $|\chi(S)| \geq 4$. Recall that we refer as an *hp-curve* in S a curve which is either an h-curve or a p-curve in S. We first show that if $a \in V_s$ is an hp-curve, then so is $\phi(a)$. Let Q be the component of S_a that is neither a handle nor a pair of pants. Superinjectivity of ϕ implies the inclusion $\phi(V_s(Q)) \subset V_s(R)$ for some component R of $S_{\phi(a)}$. It follows from Lemma 3.3 and superinjectivity of ϕ that $|\chi(R)| = |\chi(Q)|$ and $\phi(a)$ is an hp-curve.

Let $a \in V_s$ be a curve which is not an hp-curve. Let Q_1 and Q_2 be components of S_a, and let R_1 and R_2 be components of $S_{\phi(a)}$. Superinjectivity of ϕ implies that for each i, there exists j such that $\phi(V_s(Q_i)) \subset V_s(R_j)$. If the assertion (i) were not true, then $\phi(V_s(Q_1) \cup V_s(Q_2)) \subset V_s(R_j)$ for some j. By Lemma 3.3, $|\chi(R_j)| = |\chi(S)| - 1$, and thus $\phi(a)$ is an hp-curve.

Note that any simplex of $\mathcal{C}_s(S)$ contains at most $g + \lfloor p/2 \rfloor$ hp-curves if $|\chi(S)| \geq 4$. If the number of components of ∂S contained in one of components of S_a is even,
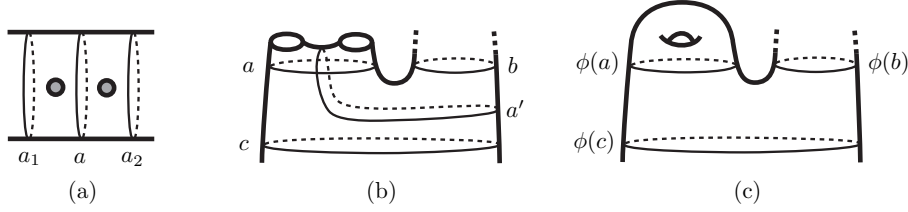
FIGURE 2.

then by choosing a simplex of $\mathcal{C}_s(S)$ containing a and $g + \lfloor p/2 \rfloor$ hp-curves, one can deduce a contradiction.

Suppose that the numbers of components of ∂S contained in both components of S_a are odd. Choose a simplex σ of maximal dimension in $\mathcal{C}_s(S)$ containing the two curves a_1 and a_2 described in Figure 2 (a). In the adjacency graph $\mathcal{G}(\sigma)$, a and a_j are adjacent for each $j = 1, 2$, and a_1 and a_2 are not adjacent. Let P denote the pair of pants in $S_{\phi(\sigma)}$ that contains the hp-curve $\phi(a)$ and is different from the one cut off by $\phi(a)$ from S. Since $\phi(a)$ and $\phi(a_j)$ are adjacent for each $j = 1, 2$ in the graph $\mathcal{G}(\phi(\sigma))$ by Lemma 3.17, $\phi(a_1)$ and $\phi(a_2)$ are boundary components of P. This is a contradiction because $\phi(a_1)$ and $\phi(a_2)$ are not adjacent in $\mathcal{G}(\phi(\sigma))$ by the same lemma.

We thus proved the assertion (i). The assertion (ii) follows from Lemma 3.3. □

Lemma 3.19. *Let $S = S_{g,p}$ be a surface with $|\chi(S)| \geq 4$. Then any superinjective map $\phi \colon \mathcal{C}_s(S) \to \mathcal{C}_s(S)$ preserves topological types, that is, Q_i and R_i are homeomorphic for each $i = 1, 2$ in the notation in Lemma 3.18.*

Proof. Once it is shown that ϕ preserves h-curves, one can easily verify that ϕ preserves topological types. To prove it, we may assume $g \geq 1$.

When $p = 0$ or 1, the claim immediately follows because ϕ is χ-preserving and there is no p-curve in S. In what follows, we assume $p \geq 2$ and that there exists a p-curve a in S such that $\phi(a)$ is an h-curve. One can find an hp-curve b in S different and disjoint from a. Lemma 3.18 implies that $\phi(b)$ is also an hp-curve. Choose $c, a' \in V_s$ as in Figure 2 (b). Since any separating curve in $S_{1,2}$ is an h-curve, $\phi(a')$ is an h-curve (see Figure 2 (c)). This contradicts the fact that ϕ is χ-preserving. □

Summarizing the argument in this section, we obtain the following:

Proposition 3.20. *Let $S = S_{g,p}$ be a surface satisfying either $g = 1$ and $p = 3$; or $g \geq 1$ and $|\chi(S)| \geq 4$. Let $\phi \colon \mathcal{T}(S) \to \mathcal{T}(S)$ be a superinjective map. Then the inclusions $\phi(V_{bp}) \subset V_{bp}$ and $\phi(V_s) \subset V_s$ hold, and the restriction of ϕ to $\mathcal{C}_s(S)$ preserves topological types.*

The latter assertion in the proposition means that for each $a \in V_s$, Q_i and R_i are homeomorphic and the inclusion $\phi(V_s(Q_i)) \subset V_s(R_i)$ holds for each $i = 1, 2$, where Q_1, Q_2 are components of S_a and R_1, R_2 are components of $S_{\phi(a)}$ with an appropriate numbering.
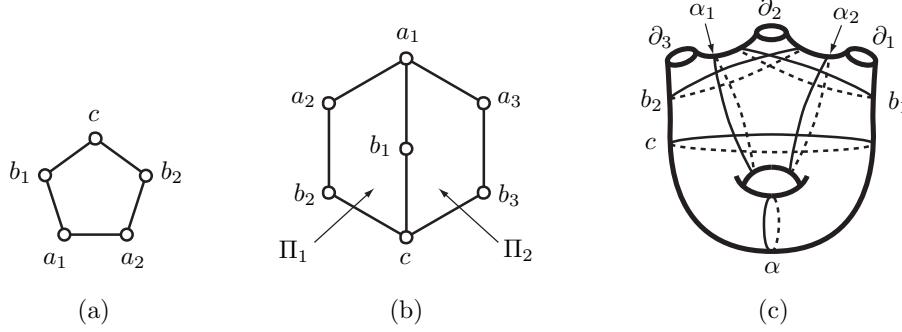
FIGURE 3.

4. SUPERINJECTIVE MAPS OF THE TORELLI COMPLEX

Let S be a surface of genus one. Given a superinjective map $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$, we construct a simplicial map $\Phi\colon \mathcal{C}(S) \to \mathcal{C}(S)$ which induces ϕ. This map Φ will be defined as follows: Let $\alpha \in V(S)$. If α is separating, then we put $\Phi(\alpha) = \phi(\alpha)$. If α is non-separating, then we choose two BPs $a, b \in V_{bp}(S)$ such that the pair $\{a, b\}$ is a rooted 1-simplex of $\mathcal{T}(S)$ whose root curve is equal to α. We then define $\Phi(\alpha) \in V(S)$ as the root of the rooted 1-simplex $\{\phi(a), \phi(b)\}$ of $\mathcal{T}(S)$. Sections 4.1 and 4.2 are devoted to showing that our construction of Φ is well-defined. Section 4.3 proves that Φ is an injective simplicial map. We then conclude that Φ is an automorphism of $\mathcal{C}(S)$ by using Theorem 2.4.

4.1. **The case $g = 1$ and $p = 3$.** We put $S = S_{1,3}$ throughout this subsection.

Lemma 4.1. *Suppose that we are given a pentagon in $\mathcal{T}(S)$ whose vertices are labeled as in Figure 3 (a). Assume that each a_i is a BP-vertex, each b_j is a p-vertex and c is an h-vertex. Let α denote the root curve for the rooted simplex $\{a_1, a_2\}$. Then $i(\alpha, b_1) = i(\alpha, b_2) = i(\alpha, c) = 0$.*

Proof. It is clear that $i(\alpha, b_1) = i(\alpha, b_2) = 0$. Note that the subsurface filled by the two p-curves b_1 and b_2 is homeomorphic to $S_{0,4}$ and contains ∂S. The h-curve c is then a boundary curve of this subsurface. Since α is disjoint from b_1 and b_2, α is disjoint from c. □

Lemma 4.2. *Let Π_1 and Π_2 be pentagons in $\mathcal{T}(S)$ sharing two edges which share a vertex as described in Figure 3 (b). Assume that each a_i is a BP-vertex, each b_j is a p-vertex and c is an h-vertex. Then the root curves for the simplices $\{a_1, a_2\}$ and $\{a_1, a_3\}$ are equal.*

Proof. Let α be the root curve for $\{a_1, a_2\}$. If the root curve for $\{a_1, a_3\}$ were not equal to α, then we could put

$$a_1 = \{\alpha, \alpha_1\}, \quad a_2 = \{\alpha, \alpha_2\}, \quad a_3 = \{\alpha_1, \alpha_3\}$$

with $\alpha_3 \neq \alpha$. Applying Lemma 4.1 to Π_1, we see $i(\alpha, c) = 0$. Since $i(a_1, c) \neq 0$, we have $i(\alpha_1, c) \neq 0$. On the other hand, applying Lemma 4.1 to Π_2, we have $i(\alpha_1, c) = 0$. This is a contradiction. □

Lemma 4.3. *Let $\alpha \in V$ be a non-separating curve in S, and let $a_1, a_2, a_3 \in V_{bp}$ be BPs such that each of the pairs $\{a_1, a_2\}$ and $\{a_1, a_3\}$ is a rooted simplex of $\mathcal{T}(S)$ whose root curve is α. Then there exists a sequence of pentagons, $\Pi_1, \ldots, \Pi_n$, in $\mathcal{T}(S)$ satisfying the following three conditions:*

> (a) *For each i, the pentagon Π_i consists of h-, p-, BP-, BP- and p-vertices in this order.*
>
> (b) *$a_2 \in \Pi_1$, $a_3 \in \Pi_n$ and $a_1 \in \Pi_i$ for each i.*
>
> (c) *For each i, the two pentagons Π_i and Π_{i+1} have two common edges which share either a BP-vertex or a p-vertex.*

Proof. We put $a_i = \{\alpha, \alpha_i\}$ for $i = 1, 2, 3$. Choose two p-curves b_1, b_2 and an h-curve c in S as described in Figure 3 (c). We then have the pentagon Π_1 in $\mathcal{T}(S)$ consisting of vertices a_1, a_2, b_2, c and b_1. Label components of ∂S as ∂_1, ∂_2 and ∂_3 as in Figure 3 (c). Let R be the component of S_{a_1} that is not a a pair of pants. It then follows that α_3 is an element of $V(R)$.

Let $h \in \mathrm{Mod}(S)$ be the half twist about b_1 exchanging ∂_1 and ∂_2, and let $x \in \mathrm{Mod}(S)$ be the BP twist about a_2. We denote by Γ the subgroup of $\mathrm{Mod}(S)$ generated by h and x. Since Γ fixes α and α_1, one obtains the natural homomorphism $p \colon \Gamma \to \mathrm{Mod}(R)$. We denote by $\mathrm{Mod}(R; \alpha, \alpha_1)$ the subgroup of $\mathrm{Mod}(R)$ consisting of all elements which fix the two components of ∂R corresponding to α and α_1. We claim that $p(\Gamma)$ is equal to $\mathrm{Mod}(R; \alpha, \alpha_1)$. Obviously, $p(h) \in \mathrm{Mod}(R)$ is the half twist about $b_1 \in V(R)$, and $p(x) \in \mathrm{Mod}(R)$ is the Dehn twist (or its inverse) about $\alpha_2 \in V(R)$. Hence, $p(\Gamma) < \mathrm{Mod}(R; \alpha, \alpha_1)$. Since the Dehn twists about b_1 and α_2 generate $\mathrm{PMod}(R)$ and since $p(h)$ exchanges ∂_1 and ∂_2, we have the inclusion $\mathrm{Mod}(R; \alpha, \alpha_1) < p(\Gamma)$. This proves the claim.

When we regard α_2 and α_3 as elements of $V(R)$, we see that α_2 and α_3 lie in the same orbit for the action of $\mathrm{Mod}(R; \alpha, \alpha_1)$ on $V(R)$ because α and α_1 are contained in different components of R_{α_2} and the same holds for R_{α_3}. The claim in the previous paragraph shows that there exist integers n_i, m_j such that

$$\alpha_3 = h^{n_1} x^{m_1} \cdots h^{n_k} x^{m_k} \alpha_2.$$

We put $s_i = n_i/|n_i|$ if $n_i \neq 0$ and put $s_i = 0$ if $n_i = 0$. Similarly, we define $t_j \in \{-1, 0, 1\}$ by using m_j. It then follows that the sequence of pentagons in $\mathcal{T}(S)$,

$$\Pi_1, \ h^{s_1}\Pi_1, \ h^{2s_1}\Pi_1, \ldots, \ h^{n_1}\Pi_1, \ h^{n_1}x^{t_1}\Pi_1, \ldots, \ h^{n_1}x^{m_1}\Pi_1, \ldots, \ h^{n_1}\cdots x^{m_k}\Pi_1,$$

satisfies the conditions (a), (b) and (c) in the lemma. $\qquad \square$

Let R be a surface of genus zero, and let ∂_1 and ∂_2 be boundary components of R. We say that a curve a in R *separates* ∂_1 and ∂_2 if ∂_1 and ∂_2 are contained in different components of R_a.

Proposition 4.4. *Let R be a surface homeomorphic to $S_{0,p}$ with $p \geq 5$, and choose two components ∂_1 and ∂_2 of ∂R. Then the full subcomplex $\mathcal{D} = \mathcal{D}(R; \partial_1, \partial_2)$ of $\mathcal{C}(R)$ spanned by all vertices corresponding to curves which separate ∂_1 and ∂_2 is connected.*

Proof. Let $\partial_1, \ldots, \partial_p$ denote components of ∂R and put $I = \{1, \ldots, p\}$. When components of ∂R is denoted as in Figure 4 (a), it is known that the family of Dehn twists t_{ij} about the simple closed curve δ_{ij} described in Figure 4 (b) for any two integers $i, j \in I$ with $2 \leq i < j \leq p$ and $1 \leq j - i \leq p - 3$ generates the
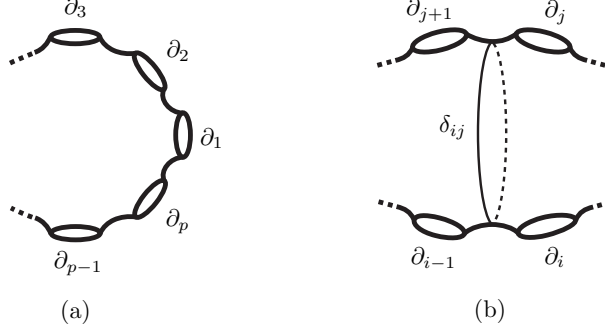
FIGURE 4.

pure mapping class group $\mathrm{PMod}(R)$ (see Chapters 1 and 4 in [7]). Let us denote by $N \subset I^2$ the set of all pairs $(i,j) \in I^2$ satisfying these two inequalities. We put $\alpha_0 = \delta_{23}$.

Given a curve a in R and a decomposition $I = I_1 \sqcup I_2$ with $|I_1|, |I_2| \geq 2$, let us say that a *decomponses* I into I_1 and I_2 if one component of R_a contains ∂_i for each $i \in I_1$ and another component of R_a contains ∂_j for each $j \in I_2$.

Claim 4.5. *Let $I = I_1 \sqcup I_2$ be a decomposition of I into two subsets such that $1 \in I_1$, $2 \in I_2$ and both I_1 and I_2 contain at least two elements. Then one can find a path in $\mathcal{D}$ connecting α_0 and a vertex α of $\mathcal{D}$ which decomposes I into I_1 and I_2.*

Proof. If $3 \in I_2$, then one can readily find a curve α such that $i(\alpha, \alpha_0) = 0$ and α decomposes I into I_1 and I_2. Assume $3 \in I_1$. If $|I_1| \geq 3$, then one can find a path of curves α_0, β, α in $\mathcal{D}$ such that β decomposes I into $I_1 \setminus \{3\}$ and $I_2 \cup \{3\}$ and α decomposes I into I_1 and I_2. If $|I_1| = 2$, then $I_1 = \{1,3\}$. One can then find a path α_0, α_1, α_2, α_3 in $\mathcal{D}$ consisting of p-vertices and satisfying the following: For each $i = 1, 2, 3$, let P_i denote the pair of pants cut off by α_i. Then P_1 contains ∂_1 and ∂_4, P_2 contains ∂_2 and ∂_5, and P_3 contains ∂_1 and ∂_3. $\square$

Claim 4.6. *There exists a path in $\mathcal{D}$ connecting α_0 and $t_{ij}(\alpha_0)$ for each $(i,j) \in N$.*

Proof. Pick $(i,j) \in N$. If $i \neq 3$, then $i(\alpha_0, \delta_{ij}) = 0$, and thus $t_{ij}(\alpha_0) = \alpha_0$. If $i = 3$ and $j \leq p-1$, then $i(\delta_{3j}, \delta_{2,p-1}) = 0$, and thus $i(t_{3j}(\alpha_0), \delta_{2,p-1}) = 0$. Since $\delta_{2,p-1}$ is a vertex of $\mathcal{D}$ and since $i(\alpha_0, \delta_{2,p-1}) = 0$, one can connect α_0 and $t_{3j}(\alpha_0)$. Figure 5 shows that α_0 and $t_{3,p}(\alpha_0)$ can be connected in $\mathcal{D}$ via α_1, α_2 and α_3. $\square$

It is easy to see that the second claim implies that each point of the orbit for the action of $\mathrm{PMod}(R)$ on $\mathcal{D}$ containing α_0 can be connected with α_0 and that the first claim then implies that $\mathcal{D}$ is connected. This kind of argument proving connectivity has already appeared in Lemma 2.1 of [31]. $\square$

Lemma 4.7. *Let $S = S_{1,p}$ be a surface with $p \geq 3$, and let $\alpha \in V$ be a non-separating curve in S. Then the full subcomplex of $\mathcal{T}(S)$ spanned by all vertices corresponding to BPs containing α is connected.*

Proof. Let R be the surface obtained by cutting S along α. We denote by ∂_1 and ∂_2 the two components of ∂R corresponding to α. It is obvious that there is a natural one-to-one correspondence between vertices of the complex $\mathcal{D} = \mathcal{D}(R; \partial_1, \partial_2)$ and BP-vertices of $\mathcal{T}(S)$ containing α. Proposition 4.4 then shows the lemma. $\square$
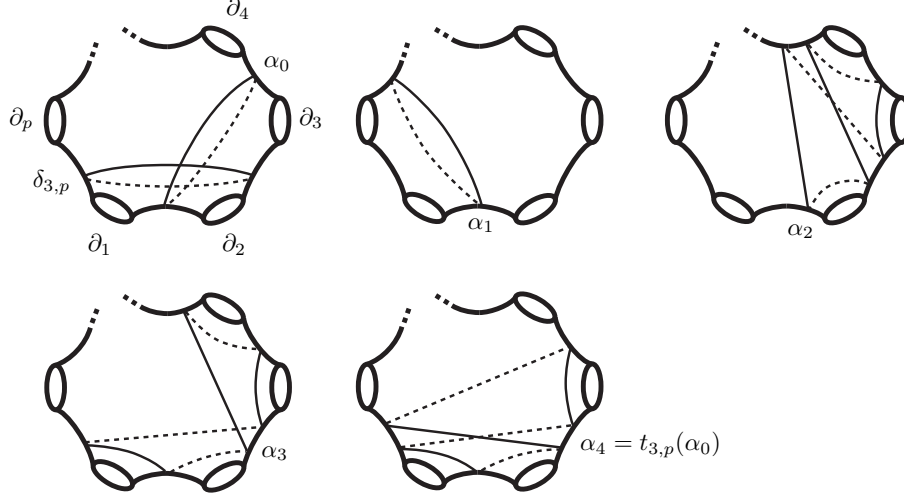
FIGURE 5.

Let $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ be a superinjective map. We define a map $\Phi\colon V(S) \to V(S)$ as follows: Let $\alpha \in V(S)$. If α is separating, then we put $\Phi(\alpha) = \phi(\alpha)$. If α is non-separating, then we choose two BPs $a, b \in V_{bp}(S)$ such that the pair $\{a, b\}$ is a rooted 1-simplex of $\mathcal{T}(S)$ whose root curve is equal to α. We then define $\Phi(\alpha) \in V(S)$ as the root curve of the rooted 1-simplex $\{\phi(a), \phi(b)\}$ of $\mathcal{T}(S)$. Lemma 3.13 implies that the pair $\{\phi(a), \phi(b)\}$ is rooted. It follows from Lemmas 4.3 and 4.7 that any two rooted 1-simplices $\{a_1, b_1\}$, $\{a_2, b_2\}$ with the same root curve α can be connected by a sequence of pentagons such that two successive pentagons in it have two common edges which share either a BP-vertex or a p-vertex. Lemma 4.2 then shows that the root curves for $\{\phi(a_1), \phi(b_1)\}$ and $\{\phi(a_2), \phi(b_2)\}$ are equal. This proves that Φ is well-defined.

4.2. **The case $g = 1$ and $p \geq 4$.** We put $S = S_{1,p}$ with $p \geq 4$ throughout this subsection.

Lemma 4.8. *The following assertions hold:*

(i) *Let b_1, b_2 and b_3 be BPs in S such that both of the pairs $\{b_1, b_2\}$ and $\{b_2, b_3\}$ form rooted simplices of $\mathcal{T}(S)$. If there exists a vertex $c \in V_t$ satisfying $i(b_1, c) = i(b_3, c) = 0$ and $i(b_2, c) \neq 0$, then the root curves of $\{b_1, b_2\}$ and $\{b_2, b_3\}$ are equal.*

(ii) *If $\sigma = \{b_1, b_2, b_3\} \in \Sigma_t(S)$ is a rooted 2-simplex, then there exists a vertex $c \in V_{bp}$ satisfying $i(b_1, c) = i(b_3, c) = 0$ and $i(b_2, c) \neq 0$.*

Proof. If the conclusion of the assertion (i) were not true, then one of the two curves in b_2 would be contained in b_1 and another curve would be contained in b_3. Hence, any curve in V intersecting b_2 intersects either b_1 or b_3. This contradicts the existence of c.

On the assumption in the assertion (ii), choose a curve $a \in V$ with $i(a, b_1) = i(a, b_3) = 0$ and $i(a, b_2) \neq 0$. The BP $c = t_a(b_2)$ satisfies the desired condition. $\square$

Lemma 4.9. *Let a, b and c be BPs in S such that both of the pairs $\{a, b\}$ and $\{b, c\}$ are rooted 1-simplices of $\mathcal{T}(S)$ and the root curves for them are equal. Let*

$\alpha \in V$ denote the root curve. Then there exists a sequence $a_1, a_2, \ldots, a_n$ of BPs in S such that for each $i = 0, 1, \ldots, n$, a_i contains α and $\{b, a_i, a_{i+1}\}$ is a 2-simplex of $\mathcal{T}(S)$, where $a_0 = a$ and $a_{n+1} = c$.

Proof. Let R be the surface obtained by cutting S along α. We denote by ∂_1 and ∂_2 the two components of ∂R corresponding to α. The surface R is homeomorphic to $S_{0,p+2}$, and we have $p + 2 \geq 6$. Note that there is a natural one-to-one correspondence between vertices of the complex $\mathcal{D} = \mathcal{D}(R; \partial_1, \partial_2)$ in Proposition 4.4 and BP-vertices of $\mathcal{T}(S)$ containing α. It therefore suffices to prove that the link of each vertex of $\mathcal{D}$ is connected. Let β be a vertex of $\mathcal{D}$. If β is not a p-curve in R, then it is clear that the link of β in $\mathcal{D}$ is connected. If β is a p-curve in R, then the component of R_β that is not a pair of pants is homeomorphic to $S_{0,p+1}$. Since we have $p + 1 \geq 5$, Proposition 4.4 implies that the link of β in $\mathcal{D}$ is connected. $\square$

Let $\phi \colon \mathcal{T}(S) \to \mathcal{T}(S)$ be a superinjective map. We define a map $\Phi \colon V(S) \to V(S)$ in the same manner as in the previous subsection. Namely, we define $\Phi = \phi$ on $V_s(S)$ and if $\alpha \in V(S)$ is a non-separating curve, then we choose two BPs $a, b \in V_{bp}(S)$ such that the pair $\{a, b\}$ is a rooted 1-simplex of $\mathcal{T}(S)$ whose root curve is equal to α. We then define $\Phi(\alpha) \in V(S)$ as the root curve for the rooted 1-simplex $\{\phi(a), \phi(b)\}$ of $\mathcal{T}(S)$. By using Lemmas 4.7 and 4.9, one can construct a sequence of rooted 2-simplices between any two given rooted 1-simplices with the same root curve α. Lemma 4.8 is then applied to proving that Φ is well-defined.

4.3. **Simplicity and injectivity.** In this subsection, we fix a surface $S = S_{1,p}$ with $p \geq 3$ and fix a superinjective map $\phi \colon \mathcal{T}(S) \to \mathcal{T}(S)$. Let $\Phi \colon V(S) \to V(S)$ be the map constructed in Sections 4.1 and 4.2. We first prove that our Φ in fact is a map inducing ϕ.

Lemma 4.10. *The equality* $\phi(\{\alpha, \beta\}) = \{\Phi(\alpha), \Phi(\beta)\}$ *holds for each BP* $\{\alpha, \beta\}$ *in* S.

Proof. It follows from the definition of Φ that both $\Phi(\alpha)$ and $\Phi(\beta)$ are contained in $\phi(\{\alpha, \beta\})$. It suffices to show $\Phi(\alpha) \neq \Phi(\beta)$. After choosing a non-separating curve γ in S such that both $\{\beta, \gamma\}$ and $\{\gamma, \alpha\}$ form BPs in S, one can readily prove $\Phi(\alpha) \neq \Phi(\beta)$ by using Lemma 3.14. $\square$

Lemma 4.11. *The map* Φ *defines a simplicial map from* $\mathcal{C}(S)$ *into* $\mathcal{C}(S)$.

Proof. Let $\alpha, \beta \in V$ be two distinct curves with $i(\alpha, \beta) = 0$. If both α and β are separating, then $i(\Phi(\alpha), \Phi(\beta)) = i(\phi(\alpha), \phi(\beta)) = 0$. If both α and β are non-separating, then $\{\alpha, \beta\}$ is a BP since S is of genus one. Lemma 4.10 shows that $\{\Phi(\alpha), \Phi(\beta)\}$ is also a BP, and thus $i(\Phi(\alpha), \Phi(\beta)) = 0$.

Suppose that α is non-separating and β is separating. Unless β is an h-curve and $\alpha \in V(H_\beta)$, where H_β is the handle cut off by β, one can find a non-separating curve $\gamma \in V$ such that $i(\beta, \gamma) = 0$ and the pair $\{\alpha, \gamma\}$ is a BP. It follows from $i(\{\alpha, \gamma\}, \beta) = 0$ that $i(\phi(\{\alpha, \gamma\}), \phi(\beta)) = 0$, and thus $i(\Phi(\alpha), \Phi(\beta)) = 0$.

We now assume that β is an h-curve and $\alpha \in V(H_\beta)$. Choose separating curves $\gamma_1, \gamma_2 \in V$ and non-separating curves $\alpha_1, \alpha_2 \in V$ as described in Figure 6. Note that β is the only h-curve in S that is disjoint from both of γ_1 and γ_2. By using Proposition 3.20, one can readily verify that the same holds for the image of these separating curves via ϕ. Since $\phi(\{\alpha, \alpha_1\}) = \{\Phi(\alpha), \Phi(\alpha_1)\}$ and $\phi(\gamma_2)$ are disjoint and since $\phi(\{\alpha, \alpha_2\}) = \{\Phi(\alpha), \Phi(\alpha_2)\}$ and $\phi(\gamma_1)$ are disjoint, we see that $\Phi(\alpha)$ is disjoint from $\phi(\gamma_1)$ and $\phi(\gamma_2)$. Therefore, $\Phi(\alpha)$ is disjoint from $\Phi(\beta)$. $\square$
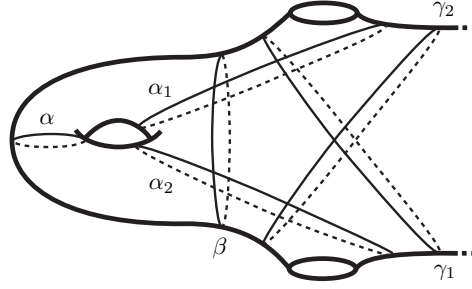
FIGURE 6.

Lemma 4.12. *The simplicial map* $\Phi\colon \mathcal{C}(S) \to \mathcal{C}(S)$ *is injective.*

Proof. Since Φ preserves separating curves and non-separating curves, respectively, and since the restriction of Φ to V_s is superinjective, it is enough to show that for non-separating curves α and β in S, the equality $\Phi(\alpha) = \Phi(\beta)$ implies $\alpha = \beta$. Note that each curve in $V(S_\alpha)$ is either separating in S or BP-equivalent to α in S. The map Φ induces a superinjective map $\Phi_\alpha\colon \mathcal{C}(S_\alpha) \to \mathcal{C}(S_{\Phi(\alpha)})$ because ϕ is superinjective. Since both S_α and $S_{\Phi(\alpha)}$ are homeomorphic to $S_{0,p+2}$, Theorem 2.4 implies that Φ_α is an isomorphism. Similarly, one obtains an isomorphism $\Phi_\beta\colon \mathcal{C}(S_\beta) \to \mathcal{C}(S_{\Phi(\beta)})$. If $\Phi(\alpha) = \Phi(\beta)$, then we have the equality

$$\phi(V_s \cap V(S_\alpha)) = \Phi(V_s \cap V(S_\alpha)) = V_s \cap V(S_{\Phi(\alpha)}) = V_s \cap V(S_{\Phi(\beta)})$$
$$= \Phi(V_s \cap V(S_\beta)) = \phi(V_s \cap V(S_\beta)).$$

Since ϕ is injective, the equality $V_s \cap V(S_\alpha) = V_s \cap V(S_\beta)$ holds. This implies $\alpha = \beta$. □

Applying Theorem 2.4, we obtain the following conclusion.

Theorem 4.13. *Let $S = S_{1,p}$ be a surface with $p \geq 3$, and let $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ be a superinjective map. Then there exists a unique automorphism Φ of $\mathcal{C}(S)$ such that we have $\Phi(\alpha) = \phi(\alpha)$ for each separating curve α in S and $\{\Phi(\beta), \Phi(\gamma)\} = \phi(\{\beta, \gamma\})$ for each BP $\{\beta, \gamma\}$ in S.*

Remark 4.14. The construction of the simplicial map $\Phi\colon \mathcal{C}(S) \to \mathcal{C}(S)$ associated with a superinjective map $\phi\colon \mathcal{T}(S) \to \mathcal{T}(S)$ is valid for a surface $S = S_{g,p}$ with $g + p \geq 5$ as well after establishing connectivity of an appropriate complex as in Proposition 4.4. However, it is not clear whether Φ is an isomorphism when $g \geq 2$. If one assumes that ϕ is an isomorphism, then it easily follows that Φ is an isomorphism.

5. AUTOMORPHISMS OF THE COMPLEX OF SEPARATING CURVES

Let $S = S_{g,p}$ be a surface. Given $\phi \in \mathrm{Aut}(\mathcal{C}_s(S))$, we construct $\Phi \in \mathrm{Aut}(\mathcal{C}(S))$ which extends ϕ. When $g = 1$, we first focus on the case $p = 3$. Results of this case is used in the case $p \geq 4$ in an inductive argument on p. When $g \geq 2$, the construction of Φ follows the argument, due to Brendle-Margalit [8], using sharing pairs and spines in S.
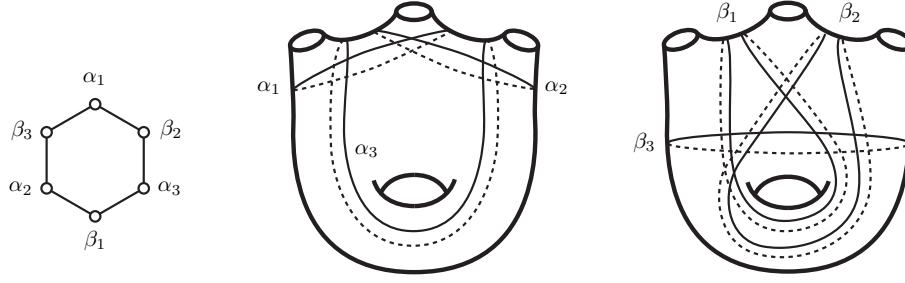
FIGURE 7. A hexagon in $\mathcal{C}_s(S_{1,3})$

5.1. **The case $g = 1$ and $p = 3$.** We put $S = S_{1,3}$ throughout this subsection. Note that the link of an h-vertex in $\mathcal{C}_s(S)$ consists of only p-vertices and that the link of a p-vertex in $\mathcal{C}_s(S)$ consists of only h-vertices. It follows that there exists no pentagon in $\mathcal{C}_s(S)$. We say that a 6-tuple $(v_1, \ldots, v_6)$ of vertices of $\mathcal{C}_s(S)$ forms a *hexagon* in $\mathcal{C}_s(S)$ if $i(v_j, v_{j+1}) = 0$, $i(v_j, v_{j+2}) \neq 0$ and $i(v_j, v_{j+3}) \neq 0$ for each j mod 6 (see Figure 7).

Lemma 5.1. *Suppose that we have a 6-tuple $(\alpha_1, \beta_3, \alpha_2, \beta_1, \alpha_3, \beta_2)$ of vertices of $\mathcal{C}_s(S)$ forming a hexagon with α_i a p-vertex for each $i = 1, 2, 3$.*

(i) *For each $i = 1, 2, 3$, let Q_i denote the component of S_{α_i} homeomorphic to $S_{1,2}$, and let ∂_i denote the component of ∂S contained in Q_i. Then ∂_1, ∂_2 and ∂_3 are different to each other.*

(ii) *For each $i = 1, 2, 3$, let R_i denote the component of S_{β_i} homeomorphic to $S_{0,4}$. Then for each i mod 3, one can realize β_{i+1} and β_{i+2} so that they are disjoint in R_i.*

(iii) *We have $i(\alpha_i, \alpha_{i+1}) = 2$ for each i mod 3.*

Proof. We first prove the assertion (i). Suppose the assertion (i) is not true. We may assume $\partial_1 = \partial_2$, and denote it by ∂. Let $\bar{S}$ be the surface obtained by filling a disk in ∂, and let $\pi \colon \mathcal{C}(S) \to \mathcal{C}^*(\bar{S})$ be the natural map, where $\mathcal{C}^*(\bar{S})$ is the simplicial cone over $\mathcal{C}(\bar{S})$ with the cone point $*$. Note that $\pi^{-1}(\{*\})$ consists of all p-curves in S cutting off a pair of pants containing ∂. It is clear that if α is a p-curve such that ∂ is contained in the component of S_α homeomorphic to $S_{1,2}$ and if β is an h-curve in S disjoint from α, then $\pi(\alpha) = \pi(\beta)$. This implies that $\pi(\beta_2) = \pi(\alpha_1) = \pi(\beta_3) = \pi(\alpha_2) = \pi(\beta_1)$. If $\partial_3 \neq \partial$, then $\pi(\beta_2) = \pi(\beta_1)$ implies $\beta_2 = \beta_1$, and this is a contradiction. Therefore, $\partial_3 = \partial$ and $\pi(\alpha_3)$ is equal to the image via π of the other five vertices. This contradicts the fact that $\pi^{-1}(\{\gamma\})$ is a simplicial tree for each $\gamma \in V(\bar{S})$ (see Section 7 of [21]).

Suppose that the assertion (ii) is not true for $i = 3$. The other cases are discussed similarly. Since α_1 is a p-curve cutting off a pair of pants containing ∂_2 and ∂_3, the intersection $\beta_2 \cap R_3$ consists of parallel arcs each of which cuts off an annulus containing ∂_1. Similarly, $\beta_1 \cap R_3$ consists of parallel arcs each of which cuts off an annulus containing ∂_2. Since $\beta_1 \cap R_3$ and $\beta_2 \cap R_3$ intersect, the union of them cuts off an annulus which contains ∂_1 and whose another boundary component consists of an arc in β_1 and an arc in β_2. This contradicts the fact that α_3 is a boundary component of a regular neighborhood of $\beta_1 \cup \beta_2$ and is a p-curve cutting off a pair of

pants containing ∂_1. This proves the assertion (ii). The assertion (iii) immediately follows from the assertion (ii). □

Lemma 5.2. *In the notation of Lemma 5.1, there exists a unique non-separating curve in S disjoint from all of the curves α_i and β_j for $i, j = 1, 2, 3$.*

Proof. Lemma 5.1 shows that each of the intersections $\alpha_2 \cap Q_1$ and $\alpha_3 \cap Q_1$ is a single separating arc in Q_1. If these two arcs were parallel, then the h-curves corresponding to boundary components of regular neighborhoods of $\alpha_1 \cup \alpha_2$ and $\alpha_1 \cup \alpha_3$ would be equal. This is a contradiction. Hence, the two arcs $\alpha_2 \cap Q_1$ and $\alpha_3 \cap Q_1$ intersect in Q_1.

Let T be the component of $S \setminus (\alpha_1 \cup \alpha_2)$ of genus one and having one boundary component. The intersection $\alpha_3 \cap T$ then consists of two parallel arcs because α_3 is separating in S. We can therefore find a unique non-separating curve c in S disjoint from α_1, α_2 and α_3. It is then obvious that c is also disjoint from β_1, β_2 and β_3. □

Given a 6-tuple $(\alpha_1, \beta_3, \alpha_2, \beta_1, \alpha_3, \beta_2)$ of vertices of $\mathcal{C}_s(S)$ forming a hexagon Π, we denote by $c(\Pi)$ the non-separating curve in S satisfying the conclusion of Lemma 5.2. The proof of Lemma 5.2 shows that the topological type of the curves α_1, α_2 and α_3 in a hexagon of $\mathcal{C}_s(S)$ is uniquely determined. We thus obtain the following:

Theorem 5.3. *The action of $\mathrm{PMod}(S)$ on the set of subcomplexes of $\mathcal{C}_s(S)$ consisting of six vertices forming a hexagon in some order is transitive.*

Lemma 5.4. *If two hexagons Π_1 and Π_2 in $\mathcal{C}_s(S)$ share two edges which share a p-vertex, then $c(\Pi_1) = c(\Pi_2)$.*

Proof. Let β_1 and β_2 be two h-vertices in a hexagon Π of $\mathcal{C}_s(S)$, and let α be the p-vertex adjacent to both of β_1 and β_2. The lemma follows from the fact that $c(\Pi)$ is the only curve in S disjoint from any of α, β_1 and β_2. □

Lemma 5.5. *Fix a non-separating curve c in S, and let Π and P be hexagons with $c(\Pi) = c(P) = c$. Then there exists a sequence of hexagons, $\Pi_1, \ldots, \Pi_n$, in $\mathcal{C}_s(S)$ satisfying the following two conditions:*

  (a) *$\Pi_1 = \Pi$ and $\Pi_n = P$.*
  (b) *For each i, the two hexagons Π_i and Π_{i+1} share two edges which share a p-vertex.*

To prove this lemma, we need the following proposition, which can be readily proved along the same spirit as Proposition 4.4.

Proposition 5.6. *We put $R = S_{0,5}$ and choose two components ∂_1, ∂_2 of ∂R. We define $\mathcal{E}$ as the simplicial graph so that*

  • *the set of vertices is given by the set of all curves in R which cut off a pair of pants containing ∂_1 and ∂_2; and*
  • *two such vertices α, β are connected by an edge if and only if $i(\alpha, \beta) = 4$.*

Then the graph $\mathcal{E}$ is connected.

Proof of Lemma 5.5. We first prove the lemma when Π and P have a common h-vertex. Let $(\alpha_1, \beta_3, \alpha_2, \beta_1, \alpha_3, \beta_2)$ be a 6-tuple of vertices of $\mathcal{C}_s(S)$ forming Π with α_i a p-vertex for each $i = 1, 2, 3$, as described in Figure 7. Assume that P contains β_3. If we denote by h the involution that is described in Figure 8 (a) and exchanges
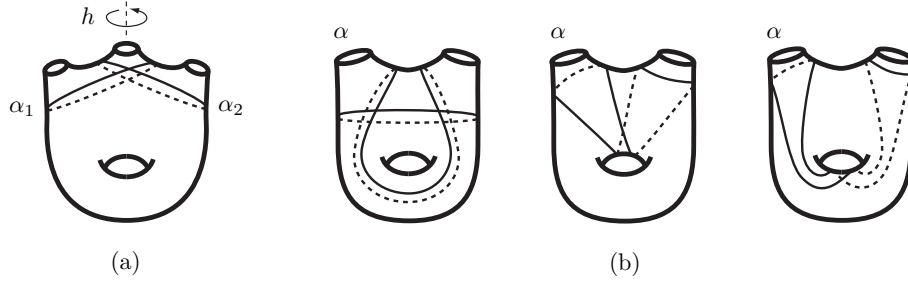
FIGURE 8. (b) Four h-curves in S any two of which are connected
by an edge in the graph $\mathcal{G}(\alpha)$

α_1 and α_2, then we have $h(\Pi) = \Pi$. Note that h and the half twists about α_1 and
α_2 generate the stabilizer of β_3 and c in $\mathrm{Mod}(S)$. The uniqueness of the topological
type of hexagons in $\mathcal{C}_s(S)$ implies that P is the image of Π by a product of h, the
half twists about α_1 and α_2 and their inverses. As in the proof of Lemma 4.3, we
can then find a sequence of hexagons satisfying the conditions (a) and (b).

In general, Proposition 5.6 shows that there exists a sequence, $a_1, a_2, \ldots, a_n$, of
vertices in $\mathcal{C}_s(S)$ such that

- $a_j \neq a_{j+1}$ and $i(a_j, a_{j+1}) = i(a_j, c) = 0$ for each j;
- a_1 is an h-vertex in Π, and a_n is an h-vertex in P; and
- for each j, if a_j is a p-vertex, then the three vertices a_{j-1}, a_j and a_{j+1} lie
  in a hexagon of $\mathcal{C}_s(S)$.

By using this sequence and the fact proved in the previous paragraph, we obtain a
desired sequence of hexagons. □

Lemma 5.7. *Any superinjective map* $\phi \colon \mathcal{C}_s(S) \to \mathcal{C}_s(S)$ *preserves h-vertices and
p-vertices in S, respectively.*

Proof. Pick $\alpha \in V_s$. We define a simplicial graph $\mathcal{G}(\alpha)$ as follows: The set of
vertices of $\mathcal{G}(\alpha)$ is given by the set of vertices in the link of α in $\mathcal{C}_s(S)$. Two
distinct vertices β_1, β_2 in $\mathcal{G}(\alpha)$ are connected by an edge if and only if there exists
a hexagon containing α, β_1 and β_2. Note that when α is an h-vertex (resp. a
p-vertex), β_1 and β_2 are connected if and only if $i(\beta_1, \beta_2) = 2$ (resp. 4).

If α is an h-vertex, then $\mathcal{G}(\alpha)$ is the Farey graph. One can easily see that if α is
a p-vertex, then $\mathcal{G}(\alpha)$ contains at least four vertices such that any two of them are
connected by an edge (see Figure 8 (b)). This proves the lemma because the Farey
graph does not contain such a subgraph and because the link of an h-vertex (resp.
a p-vertex) in $\mathcal{C}_s(S)$ consists of p-vertices (resp. h-vertices). □

Theorem 5.8. *We put $S = S_{1,3}$. Then for each automorphism ϕ of $\mathcal{C}_s(S)$, there
exists a unique automorphism Φ of $\mathcal{C}(S)$ such that $\Phi(\alpha) = \phi(\alpha)$ for each separating
curve α in S.*

Proof. Let $\phi \in \mathrm{Aut}(\mathcal{C}_s(S))$. Since ϕ preserves hexagons of $\mathcal{C}_s(S)$, we define the
extension $\Phi \colon V(S) \to V(S)$ of ϕ by putting $\Phi(c) = c(\phi(\Pi))$ for each non-separating
curve c in S, where Π is a hexagon of $\mathcal{C}_s(S)$ with $c = c(\Pi)$. This is well-defined
thanks to Lemmas 5.4, 5.5 and 5.7. It is clear that Φ is bijective.

We next prove that Φ is simplicial. It is easy to see that for two curves c_1 and c_2 in S, $i(c_1, c_2) = 0$ implies $i(\Phi(c_1), \Phi(c_2)) = 0$ unless both c_1 and c_2 are non-separating. Let c_1 and c_2 be distinct non-separating curves in S. We note that if c_1 and c_2 are disjoint, then there exist infinitely many p-curves in S disjoint from both of c_1 and c_2, and there is no h-curve in S disjoint from both of c_1 and c_2. Assume that c_1 and c_2 intersect. If the subsurface of S filled by c_1 and c_2 is homeomorphic to $S_{1,1}$, then there exists an h-curve in S disjoint from both of c_1 and c_2. Otherwise, one can find at most one p-curve in S disjoint from both of c_1 and c_2 if it exists. These observations prove that Φ preserves two disjoint non-separating curves in S and that Φ is simplicial. □

Remark 5.9. Given a superinjective map $\phi \colon \mathcal{C}_s(S) \to \mathcal{C}_s(S)$, one can define a map $\Phi \colon V(S) \to V(S)$ and prove that Φ is a simplicial map from $\mathcal{C}(S)$ into itself in the same way as in the proof of Theorem 5.8. However, it is not obvious whether Φ is (super)injective or not. Once Φ is shown to be injective, the same conclusion as Theorem 5.8 follows from Theorem 2.4.

5.2. **The case** $g = 1$ **and** $p \geq 4$. Let $S = S_{1,p}$ be a surface with $p \geq 4$, and fix $\phi \in \text{Aut}(\mathcal{C}_s(S))$. We define a simplicial automorphism of $\mathcal{C}(S)$ extending ϕ by induction on p. For an integer q with $2 \leq q \leq p$, we refer as a *q-HBC (hole bounding curve)* in S a separating curve α in S such that the component of S_α of genus zero contains exactly q components of ∂S.

Let α be a q-HBC in S with $2 \leq q \leq p-2$. By the hypothesis of the induction, we obtain a simplicial isomorphism $\phi_\alpha \colon \text{Lk}(\alpha) \to \text{Lk}(\phi(\alpha))$ extending the restriction of ϕ to $\text{Lk}(\alpha) \cap \mathcal{C}_s(S)$, where for each $\gamma \in V(S)$, $\text{Lk}(\gamma)$ denotes the link of γ in $\mathcal{C}(S)$.

We next assume that α is a $(p-1)$-HBC in S. Let Q_1 and Q_2 be two components of S_α with Q_1 of genus one. Choosing a separating curve β in Q_2, we define a simplicial isomorphism $\phi_\alpha \colon \text{Lk}(\alpha) \to \text{Lk}(\phi(\alpha))$ as $\phi_\alpha = \phi_\beta$ on $V(Q_1)$ and $\phi_\alpha = \phi$ on $V(Q_2)$. Note that β is a q-HBC with $2 \leq q \leq p-2$ and that $V(Q_2)$ is contained in $V_s(S)$. This definition is independent of the choice of γ thanks to the following lemma, which is readily proved by using Theorem 2.3.

Lemma 5.10. *We put $R = S_{1,p}$ with $p \geq 2$. Then two simplicial automorphisms of $\mathcal{C}(R)$ which preserve $V_s(R)$ and are equal on $V_s(R)$ are equal on $V(R)$.*

Let U be the set of all q-HBCs in S with $2 \leq q \leq p-1$. Lemma 5.10 also shows that if $\alpha_1, \alpha_2 \in U$ are disjoint curves, then $\phi_{\alpha_1} = \phi_{\alpha_2}$ on $\text{Lk}(\alpha_1) \cap \text{Lk}(\alpha_2)$. By using the following proposition, one obtains a simplicial automorphism Φ of $\mathcal{C}(S)$ as an extension of ϕ_α for each $\alpha \in U$.

Proposition 5.11. *We put $R = S_{0,p}$ with $p \geq 6$ and choose two components ∂_1, ∂_2 of ∂R. We define $\mathcal{F}$ as the full subcomplex of $\mathcal{C}(R)$ spanned by all curves α in R such that one component of R_α contains both ∂_1 and ∂_2 and contains at least three components of ∂R. Then $\mathcal{F}$ is connected.*

This proposition is also verified along the same idea used in the proof of Proposition 4.4. Combining Theorem 5.8, we proved the following:

Theorem 5.12. *Let $S = S_{1,p}$ be a surface with $p \geq 3$. Then for each automorphism ϕ of $\mathcal{C}_s(S)$, there exists a unique automorphism Φ of $\mathcal{C}(S)$ such that $\Phi(\alpha) = \phi(\alpha)$ for each separating curve α in S.*
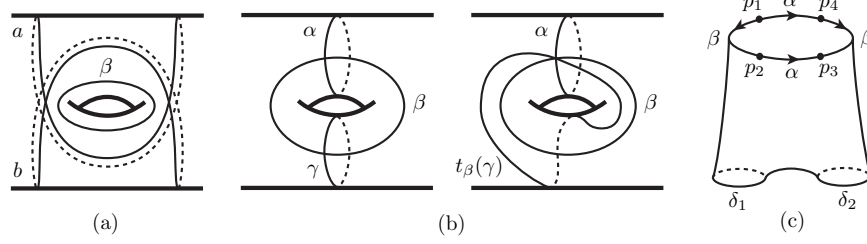
$$\text{Figure 9.}$$

5.3. **The case** $g \geq 2$. The idea for the construction of Φ due to Brendle-Margalit [8] is to use sharing pairs defined below.

Definition 5.13. Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $|\chi(S)| \geq 3$, and let a and b be h-curves in S. We denote by H_a and H_b the handles cut off by a and b, respectively. We say that a and b *share* a non-separating curve β in S if $H_a \cap H_b$ is an annulus with its core curve β and if $S \setminus (H_a \cup H_b)$ is connected (after exchanging a and b into curves isotopic to themselves if necessary). In this case, we also say that $\{a, b\}$ is a *sharing pair* for β (see Figure 9 (a)).

By looking at the arcs $b \cap H_a$, one can easily see that if a pair $\{a, b\}$ of h-curves share a non-separating curve β, then $i(a, b) = 4$, and the subsurface filled by a and b is homeomorphic to $S_{0,4}$ and has two boundary components corresponding to β. It is also shown that topological types of sharing pairs are the same, i.e., the action of $\text{PMod}(S)$ on the set of sharing pairs is transitive. Note that when S is a surface of genus one, there exists no pair $\{a, b\}$ of h-curves in S satisfying the condition in Definition 5.13. The following lemma characterizes sharing pairs in terms of disjointness and non-disjointness.

Lemma 5.14. *Let* $S = S_{g,p}$ *be a surface with* $g \geq 2$ *and* $|\chi(S)| \geq 4$, *and let* a *and* b *be h-curves in* S. *Then* a *and* b *form a sharing pair if and only if there exist separating curves* w, x, y *and* z *in* S *satisfying the following six conditions:*

- *z cuts off a subsurface Q homeomorphic to $S_{2,1}$ from S;*
- *$a, b \in V(Q)$ and $i(a, b) \neq 0$;*
- *$i(x, y) = 0$;*
- *$i(w, a) = 0$, $i(w, b) = 0$ and $i(w, z) \neq 0$;*
- *$i(x, a) \neq 0$, $i(x, b) = 0$ and $i(x, z) \neq 0$; and*
- *$i(y, a) = 0$, $i(y, b) \neq 0$ and $i(y, z) \neq 0$.*

This lemma for closed surfaces is proved in Lemma 4.1 and Addendum of [8]. The same proof of the "if" part is also valid for general surfaces. The "only if" part is proved by using Figure 10 (a) and (b) for surfaces with $g \geq 3$ and $g = 2$, respectively.

Let $\phi \in \text{Aut}(\mathcal{C}_s(S))$. Lemmas 3.19 and 5.14 imply that ϕ preserves sharing pairs. We will define a map $\Phi \colon V(S) \to V(S)$ extending ϕ by letting $\Phi(\alpha)$ be the curve shared by $\phi(a)$ and $\phi(b)$ for each non-separating curve α in S, where $\{a, b\}$ is a sharing pair for α. Spines for sharing pairs, defined below, were introduced in [8] to show that Φ is well-defined. For two curves $\alpha, \beta \in V(S)$ with $i(\alpha, \beta) = 1$, we denote by $H(\alpha, \beta)$ the handle filled by α and β.
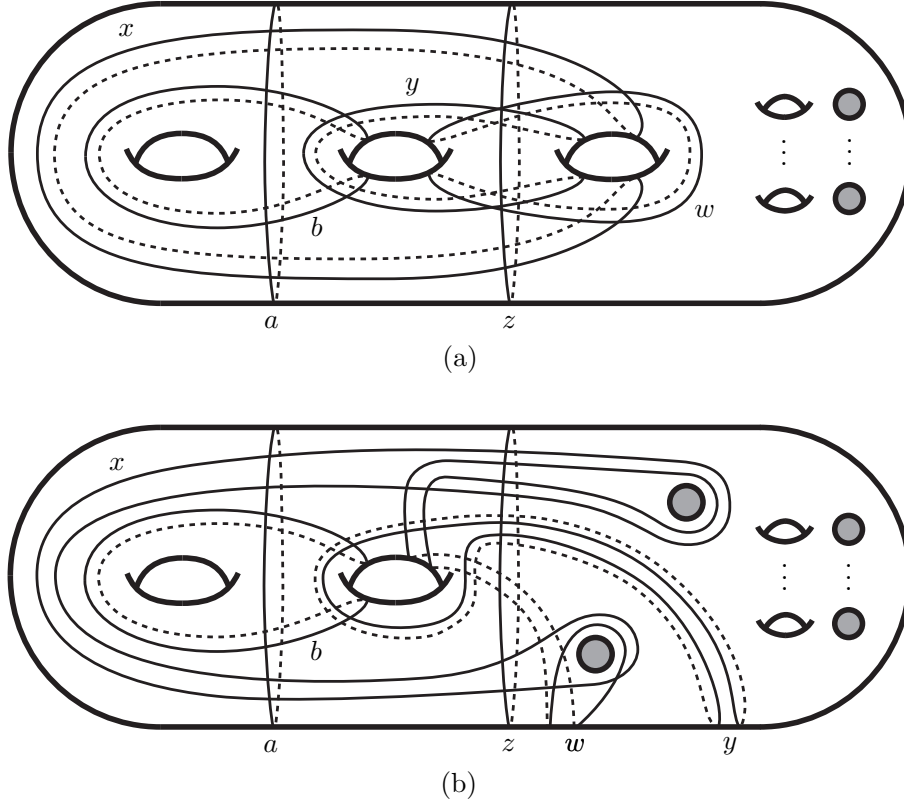
(a)

(b)

FIGURE 10.

Definition 5.15. Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $|\chi(S)| \geq 3$. A triplet of distinct non-separating curves in S, α-β-γ, is called a *spine* in S if the following three conditions are satisfied:

(a) $i(\alpha, \beta) = i(\beta, \gamma) = 1$ and $i(\alpha, \gamma) \leq 1$.
(b) Let a and b denote the boundary components of the handles $H(\alpha, \beta)$ and $H(\beta, \gamma)$, respectively. Then $\{a, b\}$ is a sharing pair for β.
(c) There exist realizations A, B and C of α, β and γ, respectively, such that they intersect transversely and $S \setminus (A \cup B \cup C)$ is connected.

In this case, α-β-γ is called a spine for the sharing pair $\{a, b\}$ (see Figure 9 (b)).

A *move* between two spines in S is defined to be a change of the form, α-β-$\gamma \mapsto \alpha$-β-γ', with γ-β-γ' a spine.

In what follows, we describe some basic properties of spines and moves between them, which will be used to prove that Φ is well-defined.

Lemma 5.16. *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $|\chi(S)| \geq 3$. Suppose that we are given a spine α-β-γ in S, and let A, B and C be realizations of α, β and γ, respectively, satisfying the condition (c) in Definition 5.15. Then*

(i) $|A \cap B| = i(\alpha, \beta) = 1$, $|B \cap C| = i(\beta, \gamma) = 1$ *and* $|C \cap A| = i(\gamma, \alpha)$.

Let R denote the surface obtained by cutting S along A and B. Let p_1, p_2, p_3 and p_4 denote the identified points on the cut end that correspond to the intersection of A and B (see Figure 9 (c)). Then

  (ii) *if $i(\gamma, \alpha) = 1$, then $|A \cap B \cap C| = 1$. In this case, C is given by an essential arc in R connecting either p_1 and p_3 or p_2 and p_4.*

  (iii) *if $i(\gamma, \alpha) = 0$, then C is given by an essential arc in R connecting a point in an arc corresponding to B with a point in another arc corresponding to B. In this case, C is isotopic to a simple closed curve in S which is given by an essential arc in R connecting either p_1 and p_4 or p_2 and p_3.*

Proof. The assertion (i) immediately follows from the Bigon Criterion (see Proposition 1.3 in [10]). Let δ_1 and δ_2 be curves in S corresponding to boundary components of the subsurface filled by α, β and γ. Since $S \setminus (A \cup B \cup C)$ is connected, C is given by a single essential arc l in R connecting either two of p_1, p_2, p_3 and p_4 or two points in the arcs corresponding to B. If l connected either p_1 and p_2 or p_3 and p_4, then C could be moved into a curve disjoint from B. This is a contradiction. The assertion (ii) and (iii) then follows. $\square$

Lemma 5.17. *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $|\chi(S)| \geq 3$. Let α-β-$\gamma \mapsto \alpha$-β-γ' be a move between two spines in S. Then the following two assertions hold:*

  (i) *Let δ_1 and δ_2 denote the boundary components of the subsurface of S filled by α, β and γ. Then γ' intersects either δ_1 or δ_2.*

  (ii) *If $|\chi(S)| \geq 4$, then there exists a separating curve δ in S which intersects γ', but is disjoint from each of α, β and γ.*

Proof. Suppose that γ' intersects nether δ_1 nor δ_2. By the definition of a move of spines, γ and γ' are given by essential arcs which connect two of p_1, p_2, p_3 and p_4 in Figure 9 (c) and are disjoint in their interiors. By using the assumption that $S \setminus (\gamma \cup \beta \cup \gamma')$ is connected, one can easily see that this is impossible. This proves the assertion (i). The assertion (ii) immediately follows from the assertion (i). $\square$

The next lemma can be proved along the same idea as Lemma 4.4 in [8] by using Lemma 5.17.

Lemma 5.18. *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $|\chi(S)| \geq 4$, and let $\phi \colon \mathcal{T}(S) \to \mathcal{T}(S)$ be a superinjective map. Suppose that we are given a move between two spines in S, α-β-$\gamma \mapsto \alpha$-β-γ'. We denote by a, b and b' the boundary components of the handles $H(\alpha, \beta)$, $H(\beta, \gamma)$ and $H(\beta, \gamma')$, respectively. Then $\{\phi(a), \phi(b)\}$ and $\{\phi(a), \phi(b')\}$ are sharing pairs for the same non-separating curve in S.*

The next proposition also follows from the argument in the proof of Proposition 4.5 in [8].

Proposition 5.19. *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $|\chi(S)| \geq 4$. For each non-separating curve β in S, any two spines α-β-γ and δ-β-ϵ differ by finitely many moves.*

Theorem 5.20. *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $|\chi(S)| \geq 4$. Then the following assertions hold:*

  (i) *For each automorphism ϕ of $\mathcal{T}(S)$, there exists a unique automorphism Φ of $\mathcal{C}(S)$ such that we have $\Phi(\alpha) = \phi(\alpha)$ for each separating curve α in S and $\{\Phi(\beta), \Phi(\gamma)\} = \phi(\{\beta, \gamma\})$ for each BP $\{\beta, \gamma\}$ in S.*
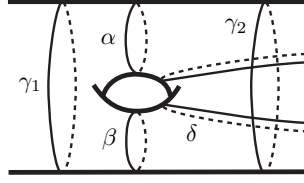
FIGURE 11.

(ii) *For each automorphism ψ of $\mathcal{C}_s(S)$, there exists a unique automorphism Ψ of $\mathcal{C}(S)$ such that $\Psi(\alpha) = \psi(\alpha)$ for each separating curve α in S.*

Proof. For each $\psi \in \mathrm{Aut}(\mathcal{C}_s(S))$, we define a map $\Psi \colon V(S) \to V(S)$ as follows: Put $\Psi(\alpha) = \psi(\alpha)$ for each separating curve α in S, and define $\Psi(\beta)$ as the non-separating curve shared by the pair $\{\psi(a), \psi(b)\}$ for each non-separating curve β in S, where $\{a, b\}$ is a sharing pair for β. This is well-defined thanks to Lemma 5.18 and Proposition 5.19. We note that Lemma 3.19 shows that ψ preserves topological types of each vertex of $\mathcal{C}_s(S)$. For an h-curve a in S, we denote by H_a the handle cut off by a. Note that in general, if a is an h-curve in S and β is a non-separating curve in H_a, then $\Psi(a)$ is also an h-curve in S, and $\Psi(\beta)$ is in the handle $H_{\Psi(a)}$. This immediately follows from the definition of the extension Ψ.

We next prove that Ψ is simplicial. Let α and β be disjoint and distinct curves in S. If both α and β are separating, then it is clear that $\Psi(\alpha)$ and $\Psi(\beta)$ are disjoint since ψ is simplicial. If α is separating and β is non-separating, then there always exists an h-curve a in S such that $i(a, \alpha) = 0$ and β is contained in H_a. Since α is either equal to a or in the complement of H_a, $\Psi(\alpha)$ and $\Psi(\beta)$ are disjoint.

Finally, we suppose that both α and β are non-separating. It is readily proved that $\Psi(\alpha)$ and $\Psi(\beta)$ are disjoint if there exist distinct and disjoint h-curves a and b such that α lies in H_a and β lies in H_b. Otherwise, α and β form a BP in S. Choose separating curves γ_1 and γ_2 in S which cut off the surface homeomorphic to $S_{1,2}$ and containing α and β (see Figure 11). Lemma 3.19 implies that $\Psi(\gamma_1)$ and $\Psi(\gamma_2)$ also cut off the surface homeomorphic to $S_{1,2}$ and containing $\Psi(\alpha)$ and $\Psi(\beta)$. Choose a separating curve δ in S which intersects exactly one of γ_1 and γ_2 and is disjoint from α and β. Suppose that $\Psi(\alpha)$ and $\Psi(\beta)$ intersect. Let Q be the subsurface filled by $\Psi(\alpha)$ and $\Psi(\beta)$. If Q is a handle, then one can deduce a contradiction by using the inverse Ψ^{-1} and the boundary curve of Q. It follows that $|\chi(Q)| = 2$ and the existence of $\Psi(\delta)$ implies a contradiction. We therefore proved that Ψ is simplicial and proved the assertion (ii).

Let $\phi \in \mathrm{Aut}(\mathcal{T}(S))$. Since ϕ preserves $V_s(S)$ by Lemma 3.7 (i), we obtain the extension $\Phi \in \mathrm{Aut}(\mathcal{C}(S))$ of the restriction of ϕ to $V_s(S)$ by using the assertion (ii). Along the same idea in Section 6 of [8] to find separating curves defining a BP, we can show the equality $\{\Phi(\beta), \Phi(\gamma)\} = \phi(\{\beta, \gamma\})$ for each BP $\{\beta, \gamma\}$ in S. $\square$

6. Twisting elements of the Torelli group

Using the result on the automorphism groups of the Torelli complex and the complex of separating curves, we compute the commensurators of the Torelli group and the Johnson kernel. We first present a few facts on abelian subgroups of the Torelli group and an algebraic characterization of Dehn twists about separating curves and BP twists.

6.1. **Abelian subgroups of the Torelli group.** The argument of this subsection heavily depends on [4], [33] and [34], where closed surfaces are dealt with. Let $S = S_{g,p}$ be a surface with $3g + p - 4 \geq 0$. Let σ be a simplex of $\mathcal{C}(S)$ if $3g + p - 4 > 0$, and let σ be a vertex of $\mathcal{C}(S)$ otherwise. Pick a curve α in σ. We say that α is of *a-type* in σ if α is separating in S. We say that α is of *b-type* in σ if α is non-separating in S and is contained in a BP-equivalence class in σ consisting of at least two curves. Otherwise, i.e., if α is non-separating in S and any other curve in σ is not BP-equivalent to α, then α is said to be of *c-type* in σ. Each curve of σ is classified into these three types. This terminology follows [33]. We denote by T_σ the subgroup of $\mathrm{Mod}(S)$ generated by Dehn twists about all curves in σ. The following theorem relies on [4] and [33].

Theorem 6.1. *Let $S = S_{g,p}$ be a surface with $3g + p - 4 \geq 0$. Let σ be a simplex of $\mathcal{C}(S)$ if $3g + p - 4 > 0$, and let σ be a vertex of $\mathcal{C}(S)$ otherwise. Then*

(i) *$T_\sigma \cap \mathcal{I}(S)$ is generated by Dehn twists about separating curves in σ and BP twists about BPs of two curves in σ.*

(ii) *$T_\sigma \cap \mathcal{K}(S)$ is generated by Dehn twists about separating curves in σ.*

Proof. When $g = 0$, the theorem immediately follows. When $(g, p) = (1, 1)$, the assertions (i) and (ii) are obvious because both $\mathcal{I}(S)$ and $\mathcal{K}(S)$ are trivial and any vertex of $\mathcal{C}(S)$ corresponds to a non-separating curve. When $g \geq 2$ and $p = 0$, the assertions (i) and (ii) are proved in Theorem 3.1 in [33] and Theorem A.1 in [4], respectively.

We prove the theorem by induction on the number of boundary components of S. We assume either $g = 1$ and $p \geq 2$ or $g \geq 2$ and $p \geq 1$. Filling a disk in a boundary component ∂_0 of S, one obtains the surface R homeomorphic to $S_{g,p-1}$ and the natural simplicial map $\pi \colon \mathcal{C}(S) \to \mathcal{C}^*(R)$, where $\mathcal{C}^*(R)$ is the simplicial cone over $\mathcal{C}(R)$ with the cone point $*$. Note that $\pi^{-1}(\{*\})$ consists of all p-curves in S cutting off a pair of pants containing ∂_0. We have the natural homomorphism $q \colon \mathrm{PMod}(S) \to \mathrm{PMod}(R)$ satisfying $q(\mathcal{I}(S)) = \mathcal{I}(R)$ and $q(\mathcal{K}(S)) = \mathcal{K}(R)$.

Let σ be a simplex of $\mathcal{C}(S)$. It is easy to see the following facts:

(a) Let b_1 and b_2 be distinct curves of b-type in σ. Note that both $\pi(b_1)$ and $\pi(b_2)$ lie in $V(R)$. If $\pi(b_1) = \pi(b_2)$, then b_1 and b_2 are BP-equivalent in S and they cut off a pair of pants containing ∂_0.

(b) If c is a curve of c-type in σ, then $\pi(c)$ is a curve of c-type in $\pi(\sigma) \setminus \{*\}$.

We can readily prove the assertions (i) by using q and the fact (b). Suppose that $x \in T_\sigma \cap \mathcal{K}(S)$ is not in the group generated by Dehn twists about all separating curves in σ. We may assume that x is a product of Dehn twists about non-separating curves in σ and their inverses. The assertion (i) implies that x is a product of BP-twists for BPs of curves in σ. By using the fact (a) and the hypothesis of the induction, we see that x is a non-zero power of the BP-twist of the BP in σ cutting off a pair of pants containing ∂_0. This contradicts Proposition 2.2. □

Lemma 6.2. *Let $S = S_{g,p}$ be a surface with $3g + p - 4 > 0$, and let σ be a simplex of $\mathcal{C}(S)$. We denote by $\nu = \nu(\sigma)$ the number of components of S_σ and denote by $\Omega = \Omega(\sigma)$ the number of components of S_σ which are neither a handle nor a pair of pants. Then*

(i) *the inequality $\mathrm{rank}(T_\sigma \cap \mathcal{I}(S)) \leq \nu - 1$ holds.*

(ii) *the inequality $\mathrm{rank}(T_\sigma \cap \mathcal{I}(S)) + \Omega \leq 2g + p - 3$ holds. If the equality holds, then each component Q of S_σ satisfies $|\chi(Q)| \leq 2$.*

(iii) *If A is an abelian reducible subgroup of $\mathcal{I}(S)$ whose canonical reduction system is equal to σ, then $\mathrm{rank}(A) \leq \mathrm{rank}(T_\sigma \cap \mathcal{I}(S)) + \Omega$.*

The assertions (i) and (ii) can be verified along the idea in the proof of Lemmas 3.1 and 3.2 in [33], respectively. The assertion (iii) immediately follows from the definition of canonical reduction systems. We refer to [15] for the definition of canonical reduction systems for subgroups of $\mathrm{Mod}(S)$. One can prove the next proposition by using Lemma 6.2 and following the proof of Theorem 3.3 in [33].

Proposition 6.3. *Let $S = S_{g,p}$ be a surface with $3g + p - 4 > 0$. If A is an abelian subgroup of $\mathcal{I}(S)$, then $\mathrm{rank}(A) \leq 2g + p - 3$ and this equality is attained for some A. The same conclusion holds for abelian subgroups of $\mathcal{K}(S)$.*

6.2. **Characterization of twisting elements.** To prove the following theorem, let us recall reduction system graphs for simplices of $\mathcal{C}(S)$, which were introduced in [33]. Let $S = S_{g,p}$ be a surface with $3g + p - 4 > 0$, and let τ be a simplex of $\mathcal{C}(S)$. The *reduction system graph* $G(\tau)$ is then defined as follows: Vertices of $G(\tau)$ are given by components of S_τ. Edges of $G(\tau)$ are given by curves in τ. The two ends of the edge corresponding to a curve c in τ are defined to be vertices corresponding to components of S_τ which lie in the left and right hand sides of c in S. Note that $G(\tau)$ may have a loop. The reader should consult [33] for basics of reduction system graphs.

Given a group Γ and an element $x \in \Gamma$, let us denote by $Z_\Gamma(x)$ the centralizer of x in Γ and denote by $Z(\Gamma)$ the center of Γ. The following result for closed surfaces is announced in [9] and is proved in [34].

Theorem 6.4. *Let $S = S_{g,p}$ be a surface with $|\chi(S)| \geq 3$ and pick $x \in \mathcal{I}(S)$. Then x is a non-zero power of either the Dehn twist about a separating curve in S or the BP twist about a BP in S if and only if the following three conditions hold:*

(a) *$Z_{\mathcal{I}(S)}(x)$ is not isomorphic to $\mathbb{Z}$;*
(b) *$Z(Z_{\mathcal{I}(S)}(x))$ is isomorphic to $\mathbb{Z}$; and*
(c) *x is contained in an abelian subgroup of $\mathcal{I}(S)$ of rank $2g + p - 3$.*

Proof. This proof follows the argument in Theorem 3.5 in [34], where closed surfaces are dealt with. As shown in Theorem A.1, each element of $\mathcal{I}(S)$ is pure in the sense of Ivanov [15]. For each $x \in \mathcal{I}(S)$, we put $Z(x) = Z_{\mathcal{I}(S)}(x)$.

If x is a non-zero power of the BP twist about an ordered BP b, then $Z(x)$ is the stabilizer of b in $\mathcal{I}(S)$. Thanks to the assumption $|\chi(S)| \geq 3$, $Z(x)$ contains a non-abelian free group. Thus, the condition (a) holds. Since each element of $Z(x)$ is pure, we have the natural homomorphism $q \colon Z(x) \to \mathcal{I}(Q) \times \mathcal{I}(R)$, where Q and R are components of S_b. One can easily see that the center of the image of q is trivial. It then follows that $Z(Z(x))$ is equal to $\ker q$, which is the cyclic group generated by t_b. This proves the condition (b). It is clear that the condition (c) holds. This argument can be also applied to the case where x is a non-zero power of the Dehn twist about a separating curve in S.

Conversely, we assume that $x \in \mathcal{I}(S)$ satisfies the conditions (a), (b) and (c). It is easy to see that x is a reducible element of infinite order. Let $\sigma \in \Sigma(S)$ be the canonical reduction system for the cyclic group generated by x. One can readily show that if σ contains a curve of either a-type or b-type, then x is a non-zero power of either the Dehn twist about a separating curve in σ or the BP twist about a BP of curves in σ.

We assume that σ consists of only curves of c-type and deduce a contradiction. By Theorem 6.1 (i), there exists a pseudo-Anosov component Q of S_σ for x. Let us denote by σ_Q the set of all curves in σ corresponding to a component of ∂Q. Let A be an abelian subgroup of $\mathcal{I}(S)$ of rank $2g + p - 3$ containing x. We denote by $\tau \in \Sigma(S)$ the canonical reduction system for A. Note that τ contains σ and that Q is a component of S_τ. By Lemma 6.2 (ii), (iii), each component of S_τ is homeomorphic to one of $S_{0,3}$, $S_{0,4}$, $S_{1,1}$ and $S_{1,2}$. If Q were homeomorphic to $S_{1,2}$, then at least one curve in σ_Q would be a curve of either a-type or b-type in σ. This is a contradiction. Therefore, Q is homeomorphic to $S_{0,4}$. Note that there is no curve in σ_Q whose both sides are contained in Q because otherwise, one of the other two curves of σ_Q would be of a-type or b-type in σ. Each curve of σ_Q is a curve of either b-type or c-type in τ. Moreover, if two curves in σ_Q are of b-type in τ, then they are not BP-equivalent in S.

Let us assume that there is a curve $c \in \sigma_Q$ which is a curve of c-type in τ. One can construct a maximal tree T in the reduction system graph $G(\tau)$ for τ containing the edge corresponding to c because the edge is not a loop. We write

$$\tau = \{b_{11}, \ldots, b_{1q_1}, b_{21}, \ldots, b_{2q_2}, \ldots, b_{p1}, \ldots, b_{pq_p}, c_1, \ldots, c_r\}$$

so that

- each b_{ij} is of b-type in τ, and the family $\{b_{i1}, \ldots, b_{iq_i}\}$ forms a BP-equivalence class in τ for each i; and
- each c_k is of c-type in τ.

Let $\nu = \nu(\tau)$ and $\Omega = \Omega(\tau)$ be the numbers defined in Lemma 6.2. Since the number of edges of T is equal to $\nu - 1$, we obtain the inequalities

$$\nu - 1 \geq \sum_{i=1}^{p}(q_i - 1) + 1 > \sum_{i=1}^{p}(q_i - 1) = \mathrm{rank}(T_\tau \cap \mathcal{I}(S))$$

and

$$\mathrm{rank}(A) \leq \mathrm{rank}(T_\tau \cap \mathcal{I}(S)) + \Omega < \nu + \Omega - 1 \leq |\chi(S)| - 1 = 2g + p - 3$$

by using Lemma 6.2. This is a contradiction.

Finally, we suppose that σ_Q consists of curves of b-type in τ. We write τ as in the previous paragraph. Note that if for each curve α in σ_Q, we choose one curve from the BP-equivalence class of α in τ and if we cut S along all chosen curves, then S is decomposed into two connected components. This implies that for any maximal tree T in $G(\tau)$, there exists a curve β in σ_Q such that T contains all edges corresponding to curves in the BP-equivalence class of β. We may assume $\beta = b_{11}$. We then obtain the inequality

$$\nu - 1 \geq q_1 + \sum_{i=2}^{p}(q_i - 1) > \sum_{i=1}^{p}(q_i - 1) = \mathrm{rank}(T_\tau \cap \mathcal{I}(S))$$

and deduce a contradiction as in the previous paragraph. $\square$

6.3. **Computation of commensurators.** The argument of this subsection has already appeared in many works to compute commensurators of mapping class groups and their subgroups (see e.g., [1], [2], [8], [12], [13], [16] and [22]). This is outlined as follows: Given an injective homomorphism between subgroups of mapping class groups, we associate a superinjective map between appropriate complexes by using characterization of twisting elements in Theorem 6.4. After proving that

the superinjective map is induced from an element of the mapping class group, we conclude that the original homomorphism is given by inner conjugation of that element.

Proposition 6.5. *Let S be a surface with $|\chi(S)| \geq 3$, and let (G, X) be either $(\mathcal{I}(S), \mathcal{T}(S))$ or $(\mathcal{K}(S), \mathcal{C}_s(S))$. Given a finite index subgroup Γ of G and an injective homomorphism $f \colon \Gamma \to G$, one can find a unique superinjective map $\phi \colon X \to X$ such that for each vertex x of X, we have*

$$f(\langle t_x \rangle \cap \Gamma) < \langle t_{\phi(x)} \rangle,$$

where $\langle t_y \rangle$ denotes the group generated by t_y for a vertex y of X. If $f(\Gamma)$ is of finite index in G, then ϕ is an isomorphism.

Proof. Let x be a vertex of X and n a non-zero integer with $t_x^n \in \Gamma$. By using Theorem 6.4 and the fact that $\mathcal{I}(S)$ is torsion-free (see Theorem A.1), one can easily conclude that $f(t_x^n)$ is a non-zero power of either the Dehn twist about a separating curve in S or the BP twist about a BP in S. If $X = \mathcal{T}(S)$, then one can define a map $\phi \colon V_t \to V_t$ so that $f(\langle t_x \rangle \cap \Gamma) < \langle t_{\phi(x)} \rangle$ for each $x \in V_t$. If $X = \mathcal{C}_s(S)$, then Proposition 2.2 implies that $f(t_x^n)$ is a non-zero power of the Dehn twist about a separating curve in S. One then defines a map $\phi \colon V_s \to V_s$ in the same way. We can prove that ϕ is simplicial and superinjective by using the fact that for two vertices x, y of X and for any non-zero integers n, m, the subgroup generated by t_x^n and t_y^m is abelian if and only if x and y are disjoint. By using f^{-1}, we can prove that ϕ is an isomorphism if $f(\Gamma)$ is of finite index in G. $\square$

Proof of Theorem 1.4. The assertion (i) follows from Theorem 4.13. Let Γ be a finite index subgroup of $\mathcal{I}(S)$ and $f \colon \Gamma \to \mathcal{I}(S)$ an injective homomorphism. Proposition 6.5 implies that there exists a superinjective map $\phi \colon \mathcal{T}(S) \to \mathcal{T}(S)$ such that for each vertex $x \in V_t$, we have $f(\langle t_x \rangle \cap \Gamma) < \langle t_{\phi(x)} \rangle$. It follows from Theorem 4.13 that ϕ is induced from an automorphism Φ of $\mathcal{C}(S)$, which is induced from a unique element $\gamma_0 \in \mathrm{Mod}^*(S)$ by Theorem 2.3. Pick $\gamma \in \Gamma$. For each separating curve a in S, we then have

$$f(\langle t_{\gamma a} \rangle \cap \Gamma) = f(\gamma \langle t_a \rangle \gamma^{-1} \cap \Gamma) = f(\gamma) f(\langle t_a \rangle \cap \Gamma) f(\gamma)^{-1}$$
$$< f(\gamma) \langle t_{\gamma_0 a} \rangle f(\gamma)^{-1} = \langle t_{f(\gamma)\gamma_0 a} \rangle,$$

and thus $\gamma_0 \gamma a = f(\gamma) \gamma_0 a$. This implies $\gamma_0 \gamma = f(\gamma) \gamma_0$. $\square$

Theorems 1.1 and 1.2 can be verified in a similar manner by using Theorems 5.12, 5.20 and Proposition 6.5.

7. Commensurators of torus braid groups

Let S be a surface and n a positive integer. Choose n base points $p_1, \ldots, p_n$ in S. We denote by $E(n, S)$ the space of embeddings of the set of n points, $\{1, \ldots, n\}$, into S with the compact-open topoloty. The *pure braid group of n-strings* on S, denoted by $PB_n(S)$, is defined to be the fundamental group $\pi_1(E(n, S))$. We refer the reader to [5], [7] and [28] for basic facts on braid groups on surfaces.

Let T denote the closed torus, and let $S = S_{1,n}$ be a surface of genus one with n boundary components. By filling disks in all components of ∂S, we obtain the Birman exact sequence

$$PB_n(T) \xrightarrow{j} \mathrm{PMod}(S) \xrightarrow{\pi} \mathrm{Mod}(T) \to 1.$$

A description of the homomorphism j shows that $\ker \pi$ is equal to $\mathcal{I}(S)$. On the other hand, $\ker j$ is equal to the center of $PB_n(T)$, denoted by Z, which is isomorphic to $\mathbb{Z}^2$ (see Proposition 4.2 in [28] for a precise description). We refer to Chapter 4 of [7] and Section 2.8 of [17] for details of the Birman exact sequence. Note that $PB_1(T)$ is isomorphic to $\pi_1(T) \simeq \mathbb{Z}^2$.

This final section describes commensurators of braid groups of n-strings on the torus T with $n \geq 2$. The following computation of commensurators of central extensions is discussed in Section 3 of [23] in a general framework, and the reader should consult the reference for more details. A description of the automorphism groups of (pure) braid groups on T is given in [35]. To carry out the computation in [23], we need the following lemma on braid groups on T.

Lemma 7.1. *For each integer $n \geq 2$, we have a homomorphism $p \colon PB_n(T) \to \mathbb{Z}^2$ such that p is injective on the center Z of $PB_n(T)$ and the image $p(Z)$ is the subgroup of $\mathbb{Z}^2$ generated by $(n, 0)$ and $(0, n)$.*

Proof. It is known that the braid group $B_n(T)$ of n-strings on T admits the following presentation (see Theorem 1.2 in [3]):

- Generators: $\sigma_1, \ldots, \sigma_{n-1}$, a, b.
- Braid relations:
(BR1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for each i, j with $|i - j| \geq 2$;
(BR2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for each i with $1 \leq i \leq n-2$;
- Mixed relations:
  (R1) $a\sigma_i = \sigma_i a$ and $b\sigma_i = \sigma_i b$ for each i with $1 < i \leq n-1$;
  (R2) $\sigma_1^{-1} a \sigma_1^{-1} a = a \sigma_1^{-1} a \sigma_1^{-1}$ and $\sigma_1^{-1} b \sigma_1^{-1} b = b \sigma_1^{-1} b \sigma_1^{-1}$;
  (R3) $\sigma_1^{-1} a \sigma_1^{-1} b = b \sigma_1^{-1} a \sigma_1$
  (TR) $[a, b^{-1}] = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1$.

One can then define a homomorphism $p \colon B_n(T) \to \mathbb{Z}^2$ by putting $p(\sigma_i) = (0, 0)$ for each i and putting $p(a) = (1, 0)$ and $p(b) = (0, 1)$. Note that $PB_n(T)$ is contained in $B_n(T)$ and that the quotient group is isomorphic to the symmetric group of n letters. It follows from geometric description of σ, a and b in Section 2.2 of [3] and of the center of $B_n(T)$ in Proposition 4.2 of [28] that the two elements

$$a(\sigma_1^{-1} a \sigma_1^{-1})(\sigma_2^{-1} \sigma_1^{-1} a \sigma_1^{-1} \sigma_2^{-1}) \cdots (\sigma_{n-1}^{-1} \sigma_{n-2}^{-1} \cdots \sigma_1^{-1} a \sigma_1^{-1} \cdots \sigma_{n-2}^{-1} \sigma_{n-1}^{-1})$$

and

$$b(\sigma_1^{-1} b \sigma_1^{-1})(\sigma_2^{-1} \sigma_1^{-1} b \sigma_1^{-1} \sigma_2^{-1}) \cdots (\sigma_{n-1}^{-1} \sigma_{n-2}^{-1} \cdots \sigma_1^{-1} b \sigma_1^{-1} \cdots \sigma_{n-2}^{-1} \sigma_{n-1}^{-1})$$

generate the center of $B_n(T)$, which is also equal to the center Z of $PB_n(T)$. It is then obvious that $p(Z)$ is generated by $(n, 0)$ and $(0, n)$. $\qquad \square$

Let $n \geq 2$ be an integer and S a surface of genus one with n boundary components. This lemma implies that the exact sequence

$$1 \to Z \to PB_n(T) \to \mathcal{I}(S) \to 1$$

virtually split. More precisely, the exact sequence

$$1 \to Z \to p^{-1}(p(Z)) \to j(p^{-1}(p(Z))) \to 1$$

splits. One can therefore find a finite index subgroup of $PB_n(T)$ isomorphic to $\mathbb{Z}^2 \times \Gamma$, where Γ is a finite index subgroup of $\mathcal{I}(S)$. By using the fact that the

center of any finite index subgroup of $\mathcal{I}(S)$ is trivial, we obtain the split exact sequence

$$1 \to \mathrm{Tv} \to \mathrm{Comm}(PB_n(T)) \to \mathrm{Comm}(\Gamma) \to 1.$$

The group Tv, called the transvection subgroup, fits into the split exact sequence

$$1 \to \varinjlim_i H^1(\Gamma_i, \mathbb{Z}^2) \to \mathrm{Tv} \to \mathrm{Comm}(\mathbb{Z}^2) \to 1,$$

where $\lim_i$ is the direct limit taken over all finite index subgroups Γ_i of Γ. As a conclusion, we have

$$\mathrm{Comm}(PB_n(T)) \simeq \mathrm{Comm}(\mathcal{I}(S_{1,n})) \ltimes (GL(2, \mathbb{Q}) \ltimes H),$$

where we put $H = \lim_i H^1(\Gamma_i, \mathbb{Z}^2)$. Note that $\mathcal{I}(S_{1,2})$ is isomorphic to $\pi_1(S_{1,1})$ and to the free group of rank two. Since we have a surjective homomorphism from $\mathcal{I}(S)$ into the free group of rank two, we see that H is isomorphic to the countably infinite dimensional vector space $\mathbb{Q}^\infty$ over $\mathbb{Q}$. Theorem 1.1 shows that $\mathrm{Comm}(\mathcal{I}(S_{1,n}))$ is naturally isomorphic to $\mathrm{Mod}^*(S_{1,n})$ when $n \geq 3$.

Appendix A. Pureness of elements of the Torelli group

The following theorem is stated in Theorem 3.1 of [14] for the pure braid group on a surface S, which is contained in the Torelli group $\mathcal{I}(S)$.

Theorem A.1. *Let $S = S_{g,p}$ be a surface with $g \geq 1$. Then each element of $\mathcal{I}(S)$ is pure. Namely, for each element $f \in \mathcal{I}(S)$, there exist $\sigma \in \Sigma(S) \cup \{\emptyset\}$ and realizations F of f and C of σ such that*

- *$F(C) = C$;*
- *F does not exchange components of C and components of the surface S_C obtained by cutting S along C; and*
- *F induces either the identity or a pseudo-Anosov homeomorphism on each component of S_C.*

Moreover, if $\tau \in \Sigma(S)$ is fixed by f, then any curve in τ and any component of S_τ are fixed by f.

It is easy to see that this theorem implies that $\mathcal{I}(S)$ is torsion-free when the genus of S is positive. The proof of this theorem will be given after the following three lemmas.

Lemma A.2. *Let $S = S_{g,p}$ be a surface with $g \geq 1$ and pick $f \in \mathcal{I}(S)$. Suppose that $\sigma \in \Sigma(S)$ is fixed by f. Choose realizations F of f and C of σ such that $F(C) = C$ and F is the identity on ∂S. Then F preserves each component of C.*

Proof. Let $\bar{S}$ denote the surface obtained by filling disks in all components of ∂S. We define the homeomorphism $\bar{F}$ of $\bar{S}$ by extending F so that $\bar{F}$ is the identity on all filled disks. We denote by $\mathcal{C}^*(\bar{S})$ the simplicial cone over $\mathcal{C}(\bar{S})$ with the cone point $*$. Let $\pi \colon \mathcal{C}(S) \to \mathcal{C}^*(\bar{S})$ be the natural map induced from the inclusion of S into $\bar{S}$, where $\pi^{-1}(\{*\})$ consists of all separating curves in S cutting off a surface of genus zero.

Assume that there are components c_1, c_2 of C with $F(c_1) = c_2$ and $c_1 \neq c_2$. We first claim that the equality $\pi([c_1]) = \pi([c_2])$ holds, where $[c]$ denotes the isotopy class of a simple closed curve c in S. When $g = 1$, the claim is obvious. Assume $g \geq 2$. Since $\bar{F}$ acts on $H_1(\bar{S}, \mathbb{Z})$ trivially, Theorem 1.2 of [15] implies that $\bar{F}$ fixes each element of $\pi(\sigma)$. Thus, we have $\pi([c_1]) = \pi([c_2])$.

If $\pi([c_1]) = \pi([c_2]) = *$, then the sets of components of ∂S contained in the surfaces of genus zero cut off by c_1 and by c_2 are different. Since $f([c_1]) = [c_2]$, this contradicts the fact that f does not exchange components of ∂S. Let us assume $\pi([c_1]) = \pi([c_2]) \in V(\bar{S})$. It then follows that c_1 and c_2 cut off a holed annulus A from S. Since f does not exchange components of ∂S, one sees $F(A) = A$. Orient c_1 and c_2 so that they are parallel when disks are filled in all components of $A \cap \partial S$. It follows from $F(c_1) = c_2$ and $F(A) = A$ that the orientations of $F(c_1)$ and c_2 are different. If c_1 is non-separating in S, then so is c_2, and both c_1 and c_2 are non-zero as an element of $H_1(\bar{S}, \mathbb{Z})$. This contradicts the fact that $\bar{F}$ acts on $H_1(\bar{S}, \mathbb{Z})$ trivially. If c_1 is separating in S, then so is c_2. For each $i = 1, 2$, let R_i be the component of S_{c_i} that does not contain A. We then have $F(R_1) = R_2$. In particular, R_1 and R_2 are homeomorphic and are of positive genus. This also contradicts the fact that $\bar{F}$ acts on $H_1(\bar{S}, \mathbb{Z})$ trivially. $\square$

Lemma A.3. *In the notation of Lemma A.2, F preserves an orientation of each component of C. Moreover, F preserves each component of S_C.*

Proof. Let c be a component of C. If c is non-separating in S, then one can prove that F preserves an orientation of c by using the fact that $\bar{F}$ acts on $H_1(\bar{S}, \mathbb{Z})$ trivially, as in the proof of Lemma A.2. Suppose that c is separating in S. If F reversed an orientation of c, then F would exchange the two components of S_c. One can then deduce a contradiction as in the proof of Lemma A.2. The latter assertion of the lemma follows from Lemma A.2 and the former assertion. $\square$

Lemma A.4. *Let $S = S_{g,p}$ be a surface with $g \geq 1$, and pick $f \in \mathcal{I}(S)$ and $\sigma \in \Sigma(S) \cup \{\emptyset\}$ with $f\sigma = \sigma$. Choose realizations F of f and C of σ such that $F(C) = C$ and F is the identity on ∂S. Let Q be a component of S_C, and suppose that the mapping class of the homeomorphism F_Q on Q induced from F is of finite order as an element of $\mathrm{PMod}(Q)$. Then F_Q is isotopic to the identity.*

Proof. We prove this lemma by induction on p, the number of components of ∂S. When $p = 0, 1$, the lemma follows from Lemma 1.6 of [15] because f acts on $H_1(S, \mathbb{Z})$ trivially. Assume $p \geq 2$, and let Q be a component of S_C. If Q is a pair of pants, then it is clear that F_Q is isotopic to the identity. We thus assume that Q is not a pair of pants.

We first assume that Q contains a component of ∂S. By filling a disk D_1 in that component of ∂S, one obtains the surfaces $Q_1 = Q \cup D_1$ and $S_1 = S \cup D_1$ with $\chi(Q_1) < 0$ since Q is not a pair of pants. Let F_1 be the homeomorphism of S_1 defined by the extention of F that is the identity on D_1. Note that C determines a simplex of $\mathcal{C}(S_1)$ and that $p_1(f)$ belongs to $\mathcal{I}(S_1)$, where $p_1 \colon \mathrm{PMod}(S) \to \mathrm{PMod}(S_1)$ is the natural homomorphism. Since the mapping class, denoted by f_Q, of F_Q is of finite order, so is the mapping class, denoted by f_{Q_1}, of the restriction of F_1 to Q_1. The hypothesis of the induction implies that f_{Q_1} is the identity. It then follows that f_Q lies in the kernel of the natural homomorphism from $\mathrm{PMod}(Q)$ into $\mathrm{PMod}(Q_1)$, which is isomorphic to $\pi_1(Q_1)$ and is torsion-free. Therefore, f_Q is the identity.

We next assume that Q contains no component of ∂S. By filling a disk D_2 in a component ∂ of ∂S, we obtain the surface $S_2 = S \cup D_2$. It is then possible either that there are two components of C which are isotopic in S_2 to each other or that there is a component of C which is isotopic in S_2 to a component of ∂S_2. If the former is the case, then delete one of those two components of C. If the latter is

the case, then delete that component of C. Otherwise, we do nothing. We then obtain the family, denoted by C_0, of essential simple closed curves in S_2 which are pairwise non-isotopic in S_2. We denote by Q_0 the component of S_{C_0} containing Q. Note that the complement of the interior of Q in Q_0 is an annulus containing D_2 if it is non-empty. Let F_2 denote the homemomorphism of S_2 defined by the extension of F which is the identity on D_2. Since the mapping class of F_Q is of finite order, so is the mapping class of the restriction of F_2 to Q_0, denoted by F_{Q_0}. By the hypothesis of the induction, F_{Q_0} is isotopic to the identity, and thus F_Q is isotopic to the identity. $\square$

Proof of Theorem A.1. The latter assertion of the theorem follows from Lemmas A.2 and A.3. Let C be a realization of the canonical reduction system $\sigma \in \Sigma(S) \cup \{\emptyset\}$ for the cyclic group generated by f with $F(C) = C$. We may assume that F is the identity on ∂S. Lemmas A.2 and A.3 imply that F preserves each component of C and each component of S_C. It follows from Theorem 7.16 in [15] that the mapping class of the restriction of F to each component of S_C is either of finite order or pseudo-Anosov. By Lemma A.4, the restriction is either trivial or pseudo-Anosov. $\square$

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